Automorphy for Some \(l\)-Adic Lifts of Automorphic Mod \(l\) Galois Representations

AUTOMORPHY FOR SOME l-ADIC LIFTS OF AUTOMORPHIC MOD l GALOIS REPRESENTATIONS.

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ABSTRACT

We extend the methods of Wiles and of Taylor and Wiles from GL_2 to higher rank unitary groups and establish the automorphy of suitable conjugate self-dual, regular (de Rham with distinct Hodge-Tate numbers), minimally ramified, l-adic lifts of certain automorphic mod l Galois representations of any dimension. We also make a conjecture about the structure of mod l automorphic forms on definite unitary groups, which would generalise a lemma of Ihara for GL_2. Following Wiles' method we show that this conjecture implies that our automorphy lifting theorem could be extended to cover lifts that are not minimally ramified.

1. Introduction

In this paper we discuss the extension of the methods of Wiles [W] and Taylor-Wiles [TW] from GL_2 to unitary groups of any rank.

The method of [TW] does not extend to GL_n as the basic numerical coincidence on which the method depends (see corollary 2.43 and theorem 4.49 of [DDT]) breaks down. For the Taylor-Wiles method to work when considering a representation

$$r : \mathrm{Gal}\,(\overline{F}/F) \hookrightarrow G(\overline{\mathbf{Q}}_l)$$

one needs

$$[F:\mathbf{Q}](\dim G - \dim B) = \sum_{v|\infty} H^0(\mathrm{Gal}\,(\overline{F}_v/F_v), \mathrm{ad}\,^0\overline{r})$$

where B denotes a Borel subgroup of a (not necessarily connected) reductive group G and $\mathrm{ad}\,^0$ denotes the kernel of the map, $\mathrm{ad} \to \mathrm{ad}_G$, from ad to its G-coinvariants. This is an 'oddness' condition, which can only hold if F is totally real (or $\mathrm{ad}\,^0 = (0)$) and $\overline{r}$ satisfies some sort of self-duality. For instance one can expect positive results if $G = GSp_{2n}$ or $G = GO(n)$, but not if $G = GL_n$ for $n > 2$.

In this paper we work with a disconnected group $\mathcal{G}_n$ which we define to be the semidirect product of $GL_n \times GL_1$ by the two element group $\{1, \jmath\}$

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with
$$\jmath(g,\mu)\jmath^{-1} = (\mu\,{}^t g^{-1}, \mu).$$
The advantage of this group is its close connection to GL_n and the fact that Galois representations valued in the l-adic points of this group should be connected to automorphic forms on unitary groups, which are already quite well understood. This choice can give us information about certain Galois representations
$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l),$$
where F is a CM field. If c denotes complex conjugation then the representations r which arise all have the following property: There is a non-degenerate symmetric pairing $\langle\ ,\ \rangle$ on $\overline{\mathbf{Q}}_l^n$ and a character $\chi : \mathrm{Gal}\,(\overline{F}/F) \to \overline{\mathbf{Q}}_l^\times$ such that
$$\langle \sigma x, c\sigma c^{-1} y\rangle = \chi(\sigma)\langle x, y\rangle$$
for all $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$. Let F^+ denote the maximal totally real subfield of F. By restriction this also gives us information about Galois representations
$$r : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$
for which there is a non-degenerate bilinear form $(\ ,\ )$ on $\overline{\mathbf{Q}}_l^n$ and a character $\chi : \mathrm{Gal}\,(\overline{F}/F^+) \to \overline{\mathbf{Q}}_l^\times$ such that
$$(y, x) = \chi(c)(x, y)$$
and
$$(\sigma x, \sigma y) = \chi(\sigma)(x, y)$$
for all $\sigma \in \mathrm{Gal}\,(\overline{F}/F^+)$.

In this setting the Taylor-Wiles argument carries over well, and we are able to prove $R = \mathbf{T}$ theorems in the 'minimal' case. Here, as usual, R denotes a universal deformation ring for certain Galois representations and $\mathbf{T}$ denotes a Hecke algebra for a definite unitary group. By 'minimal' case, we mean that we consider deformation problems where the lifts on the inertia groups away from l are completely prescribed. (This is often achieved by making them as unramified as possible, hence the word 'minimal'.) That this is possible may come as no surprise to experts. The key insights that allow this to work are already in the literature:

1. The discovery by Diamond [Dia] and Fujiwara that Mazur's 'multiplicity one principle' (or better 'freeness principle' - it states that a certain natural module for a Hecke algebra is free) was not needed for the Taylor-Wiles argument. In fact they show how the Taylor-Wiles argument can be improved to give a new proof of this principle.

2. The discovery by Skinner and Wiles [SW] of a beautiful trick using base change to avoid the use of Ribet's 'lowering the level' results.

3. The proof of the local Langlands conjecture for GL_n and its compatibility with the instances of the global correspondence studied by Kottwitz and Clozel. (See [HT].)

Indeed a preliminary version of this manuscript has been available for many years. One of us (R.T.) apologises for the delay in producing the final version.

We will now state a sample of the sort of theorem we prove. (See corollary 4.4.4.)

Theorem A *Let $n \in \mathbf{Z}_{\geq 1}$ be even and let $l > \max\{3, n\}$ be a prime. Let S be a finite non-empty set of rational primes such that if $q \in S$ then $q \neq l$ and $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n$. Also let*

$$r : \mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow GSp_n(\mathbf{Z}_l)$$

be a continuous irreducible representation with the following properties.

1. r ramifies at only finitely many primes.

2. $r|_{\mathrm{Gal}\,(\overline{\mathbf{Q}}_l/\mathbf{Q}_l)}$ is crystalline.

3. $\dim_{\mathbf{Q}_l} \mathrm{gr}^i(r \otimes_{\mathbf{Q}_l} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{\mathbf{Q}}_l/\mathbf{Q}_l)} = 0$ unless $i \in \{0, 1, ..., n-1\}$ in which case it has dimension 1.

4. If $q \in S$ then $r|^{\mathrm{ss}}_{G_{\mathbf{Q}_q}}$ is unramified and $r|^{\mathrm{ss}}_{G_{\mathbf{Q}_q}}(\mathrm{Frob}_q)$ has eigenvalues $\{\alpha q^i : \ i = 0, 1, ..., n-1\}$ for some α.

5. If $p \notin S \cup \{l\}$ is a prime then $r(I_{\mathbf{Q}_p})$ is finite.

6. The image of $r \bmod l$ contains $Sp_n(\mathbf{F}_l)$.

7. $r \bmod l$ arises from a cuspidal automorphic representation π_0 of $GL_n(\mathbf{A})$ for which $\pi_{0,\infty}$ has trivial infinitesimal character and, for all $q \in S$ the representation $\pi_{0,q}$ is an unramified twist of the Steinberg representation.

Then r arises from a cuspidal automorphic representation π of $GL_n(\mathbf{A})$ for which π_∞ has trivial infinitesimal character and π_q is an unramified twist of the Steinberg representation.

We also remark that we actually prove a more general theorem which among other things allows one to work over any totally real field, and with any weight which is small compared to l, and with $\overline{r}$ with quite general image. (See theorems 4.4.2 and 4.4.3.)

Let us comment on the conditions of this theorem. The sixth condition is used to make the Cebotarev argument in the Taylor-Wiles method work. Much weaker conditions are possible. (See theorem 4.4.3.) One expects to need to assume that r is de Rham at l. The stronger assumption that it be crystalline and that the Hodge-Tate numbers lie in a range which is small compared to l is imposed so that one can use the theory of Fontaine and Laffaille to calculate the relevant local deformation ring at l. The assumptions that r is valued in the symplectic group and that the Hodge-Tate numbers are different are needed so that the numerology behind the Taylor-Wiles method works out. This is probably essential to the method. The condition on $r|_{G_{\mathbf{Q}_q}}$ for $q \in S$ says that the representation looks as if it could correspond under the local Langlands correspondence to a Steinberg representation. The set S needs to be non-empty so that we can transfer the relevant automorphic forms to and from unitary groups and so that we can attach Galois representations to them. As the trace formula technology improves one may be able to relax this condition. The condition that $r(I_{\mathbf{Q}_p})$ is finite for $p \notin S \cup \{l\}$ reflects the fact that we are working in the minimal case. It is a very serious restriction and seems to make this theorem nearly useless for applications.

Our main aim in this paper was to remove this minimality condition. Our strategy was to follow the arguments of Wiles in [W]. We were not able to succeed in this. Rather we were able to reduce the non-minimal case to an explicit conjecture about mod l modular forms on unitary groups, which generalises Ihara's lemma on elliptic modular forms. We will explain this more precisely in a moment. After we had made this paper public one of us (R.T.) found a new approach to the non-minimal case, which bypasses Wiles' level raising arguments and treats the minimal and non-minimal cases simultaneously using a form of the Taylor-Wiles argument. Thus in some sense this part of the present paper has been superseded by [Tay]. However we still believe that our present arguments have some value. For one thing they would prove a stronger result. In [Tay] a Hecke algebra is identified with a universal deformation ring *modulo its nilradical.* This does not suffice for special value formulae for the associated adjoint L-function. However the method of the present paper would provide this more detailed information and prove that the relevant universal deformation ring is a complete intersection, *if* one assumes our conjectural generalisation of Ihara's lemma. In addition we believe that our conjectural generalisation of Ihara's lemma may prove important in the further study of arithmetic automorphic forms on unitary groups.

To describe this conjecture we need some notation. Let F^+ be a totally real field and let G/F^+ be a unitary group with $G(F^+_\infty)$ compact. Then G becomes an inner form of GL_n over some totally imaginary quadratic extension F/F^+. Let v be a place of F^+ with $G(F^+_v) \cong GL_n(F^+_v)$ and consider an open compact subgroup $U = \prod_{w \not| v\infty} U_w \subset G(\mathbf{A}^{\infty,v}_{F^+})$. Let l be a prime not divisible by v. Then we will consider the space $\mathcal{A}(U,\overline{\mathbf{F}}_l)$ of locally constant functions

$$G(F^+)\backslash G(\mathbf{A}_{F^+}^\infty)/U \longrightarrow \overline{\mathbf{F}}_l.$$

It is naturally an admissible representation of $GL_n(F_v^+)$ and of the commutative Hecke algebra

$$\mathbf{T} = \mathrm{Im}\,({\bigotimes_w}' \overline{\mathbf{F}}_l[U_w\backslash G(F_w^+)/U_w] \longrightarrow \mathrm{End}\,(\mathcal{A}(U,\overline{\mathbf{F}}_l)),$$

with the restricted tensor product taken over places $w \neq v$ for which the isomorphism between $G(F_w^+)$ and $GL_n(F_w^+)$ identified U_w with $GL_n(\mathcal{O}_{F^+,w})$. Subject to some minor restrictions on G we can define what it means for a maximal ideal $\mathfrak{m}$ of $\mathbf{T}$ in the support of $\mathcal{A}(U,\overline{\mathbf{F}}_l)$ to be *Eisenstein* - the associated $\bmod\, l$ Galois representation of $\mathrm{Gal}\,(\overline{F}/F)$ should be reducible. (See section 3.4 for details.) Then we conjecture the following.

Conjecture B *For any F^+, G, U, v and l as above, and for any irreducible $G(F_v^+)$-submodule*

$$\pi \subset \mathcal{A}(U,\overline{\mathbf{F}}_l)$$

either π is generic or it has an Eisenstein prime of $\mathbf{T}$ in its support.

In fact a slightly weaker statement would suffice for our purposes. See section 5.3 for details. For rank 2 unitary groups this conjecture follows from the strong approximation theorem. There is another argument which uses the geometry of quotients of the Drinfeld upper half plane. An analogous statement for $GL_2/\mathbf{Q}$ is equivalent to Ihara's lemma (lemma 3.2 of [I]). This can be proved in two ways. Ihara deduced it from the congruence subgroup property for $SL_2(\mathbf{Z}[1/v])$. Diamond and Taylor [DT] found an arithmetic algebraic geometry argument. The case of GL_2 seems to be unusually easy as non-generic irreducible representations of $GL_2(F_v^+)$ are one dimensional. We have some partial results when $n = 3$, to which we hope to return in a future paper. We stress the word 'submodule' in the conjecture. The conjecture is not true for 'subquotients'. The corresponding conjecture is often known to be true in characteristic 0, where one can use trace formula arguments to compare with GL_n. (See section 5.3 for more details.)

An example of what we can prove assuming this conjecture is the following strengthening of theorem A.

Theorem C *If we assume conjecture B then theorem A remains true without the assumption 5.*

We remark that to prove this theorem we need conjecture B not just for unitary groups defined over $\mathbf{Q}$, but also over other totally real fields.

We go to considerable length to prove a similar theorem where instead of assuming that $\bar{r}$ is automorphic one can assume that it is induced from a character. (See theorems 5.6.1 and 5.6.2.) Along the way to the proof of these latter theorems we prove an analogue of Ramakrishna's lifting theorem [Ra2] for $\mathcal{G}_n$. (See theorem 2.6.3 and, for a simple special case which may be easier to appreciate, corollary 2.6.4.)

One of the problems in writing this paper has been to decide exactly what generality to work in. We could certainly have worked in greater generality, but in the interests of clarity we have usually worked in the minimal generality which we believe will be useful. In particular we have restricted ourselves to the 'crystalline' case. It would be useful, and not very difficult, to include also the ordinary case. It would also be useful to clarify the more general results that are available in the case $n = 2$.

In the first chapter of this paper we discuss deformation theory and Galois theory. We set up the Galois theoretic machinery needed for the Taylor-Wiles method (see proposition 2.5.9) and also take the opportunity to give an analogue (see theorem 2.6.3 and corollary 2.6.4) of Ramakrishna's lifting theorem [Ra2] for $\mathcal{G}_n$. In the last section of this chapter we go to considerable lengths to prove a version of this lifting theorem when the $\bmod l$ representation we are lifting is induced from a character of a cyclic extension. This strengthening is needed to prove modularity lifting theorems for these same $\bmod l$ representations. (It will be used to construct a lift whose restriction to some decomposition group corresponds, under the local Langlands correspondence, to a Steinberg representation.) This chapter was originally written in the language of deformation rings, but at the referees' suggestion we have rewritten it in Kisin's language of framed deformation rings to make it easier to read in conjunction with [Tay].

In the second chapter we discuss automorphic forms on definite unitary groups, their associated Hecke algebras, their associated Galois representations and results about congruences between such automorphic forms. In the final section of this chapter we put these results together to prove an $R = T$ theorem in the minimal case (see theorem 3.5.1). In the third chapter we use base change arguments to deduce (minimal) modularity lifting theorems for GL_n (see theorems 4.4.2 and 4.4.3).

In the final chapter we discuss our conjectural generalisation of Ihara's lemma (conjecture I), and explain how it would imply a non-minimal $R = \mathbf{T}$ theorem (theorem 5.4.1) and non-minimal modularity lifting theorems (see theorems 5.5.1 and 5.5.2). In the last section we explain how to generalise these theorems to some cases where the residual representation has a small image in the sense that it is induced from a character. This is where we use the last section of chapter one. Some of the results in this chapter depend on previously unpublished work of Marie-France Vignéras and of Russ Mann. Marie-France has kindly written up her results in an appendix to this paper. Russ has left academia and as it seems unlikely that he will ever fully write up his results (see [Man2]) we have included an account of his work in another appendix.

For the reader interested only in the main results of [Tay] and [HSBT], there is no need to read chapter 4 or the appendices of this paper. These other papers do not depend on them. (There is also no need to read sections 3.5 and 4.4.)

Finally we would like to express our great gratitude to the referees who did a wonderful job. This paper is not only more accurate, but also (we believe) much more readable thanks to their efforts.

2. Galois deformation rings.

2.1. *Some algebra.* — As explained in the introduction we are going to be concerned with homomorphisms from Galois groups to a certain disconnected group $\mathcal{G}_n$. In this section we define $\mathcal{G}_n$ and make a general study of homomorphisms from other groups to $\mathcal{G}_n$.

For n a positive integer let $\mathcal{G}_n$ denote the group scheme over $\mathbf{Z}$ which is the semi-direct product of $GL_n \times GL_1$ by the group $\{1, \jmath\}$ acting on $GL_n \times GL_1$ by

$$\jmath(g,\mu)\jmath^{-1} = (\mu\, {}^t g^{-1}, \mu).$$

(If x is a matrix we write ${}^t x$ for its transpose.) There is a homomorphism $\nu : \mathcal{G}_n \to GL_1$ which sends (g,μ) to μ and $\jmath$ to -1. Let $\mathcal{G}_n^0$ denote the connected component of $\mathcal{G}_n$. Let $\mathfrak{g}_n$ denote $\operatorname{Lie} GL_n \subset \operatorname{Lie} \mathcal{G}_n$ and ad the adjoint action of $\mathcal{G}_n$ on $\mathfrak{g}_n$. Thus for $x \in \mathfrak{g}$ we have

$$(\operatorname{ad}(g,\mu))(x) = gxg^{-1}$$

and

$$(\operatorname{ad}(\jmath))(x) = -{}^t x.$$

We also write $\mathfrak{g}_n^0$ for the subspace of $\mathfrak{g}_n$ consisting of elements of trace zero. Over $\mathbf{Z}[1/2]$ we have

$$\mathfrak{g}_n^{\mathcal{G}_n} = (0).$$

Suppose that Γ is a group, that Δ is a subgroup of index 2. Whenever we endow Γ with a topology we will assume that Δ is closed (and hence also open).

Lemma 2.1.1 *Suppose that R is a ring and $\gamma_0 \in \Gamma - \Delta$. Then there is a natural bijection between the following two sets.*

1. *Homomorphisms $r : \Gamma \to \mathcal{G}_n(R)$ that induce isomorphisms $\Gamma/\Delta \xrightarrow{\sim} \mathcal{G}_n/\mathcal{G}_n^0$.*
2. *Triples $(\rho, \mu, \langle\ ,\ \rangle)$, where $\rho : \Delta \to GL_n(R)$ and $\mu : \Gamma \to R^\times$ are homomorphisms and*

$$\langle\ ,\ \rangle : R^n \times R^n \longrightarrow R$$

is a perfect R linear pairing such that for all $x, y \in R^n$ and all $\delta \in \Delta$ we have

- *$\langle x, \rho(\gamma_0^2) y\rangle = -\mu(\gamma_0)\langle y, x\rangle$, and*
- *$\mu(\delta)\langle x, y\rangle = \langle \rho(\delta)x, \rho(\gamma_0\delta\gamma_0^{-1})y\rangle$.*

Under this correspondence $\mu(\gamma) = (\nu \circ r)(\gamma)$ *for all* $\gamma \in \Gamma$*, and*

$$\langle x, y \rangle = {}^t x A^{-1} y,$$

where $r(\gamma_0) = (A, -\mu(\gamma_0))\jmath$*. If* Γ *and* R *have topologies then under this correspondence continuous* r*'s correspond to continuous* ρ*'s.*

Note that in the special case $\gamma_0^2 = 1$ the pairing $\langle\ ,\ \rangle$ is either symmetric or anti-symmetric.

If $r : \Gamma \to \mathcal{G}_n(R)$, it will sometimes be convenient to abuse notation and also use r to denote the homomorphism $\Delta \to GL_n(R)$ obtained by composing the restriction of r to Δ with the natural projection $\mathcal{G}_n^0 \to GL_n$.

Lemma 2.1.2 *Suppose that* R *is a ring and that* $(\ ,\ )$ *is a perfect bilinear pairing* $R^n \times R^n \to R$*, which satisfies*

$$(x, y) = (-1)^a (y, x).$$

Say

$$(x, y) = {}^t x J y$$

for $J \in M_n(R)$*. Let* $\delta_{\Gamma/\Delta} : \Gamma/\Delta \xrightarrow{\sim} \{\pm 1\}$*. Suppose that* $\mu : \Gamma \to R^\times$ *and* $\rho : \Gamma \to GL_n(R)$ *are homomorphisms satisfying*

$$(\rho(\gamma)x, \rho(\gamma)y) = \mu(\gamma)(x, y)$$

for all $\gamma \in \Gamma$ *and* $x, y \in R^n$*. Then there is a homomorphism*

$$r : \Gamma \longrightarrow \mathcal{G}_n(R)$$

defined by

$$r(\delta) = (\rho(\delta), \mu(\delta))$$

if $\delta \in \Delta$*, and*

$$r(\gamma) = (\rho(\gamma) J^{-1}, (-1)^a \mu(\gamma))\jmath$$

if $\gamma \in \Gamma - \Delta$*. Moreover*

$$\nu \circ r = \delta_{\Gamma/\Delta}^{a+1} \mu.$$

Let us introduce induction in this setting. Suppose that Γ' is a finite index subgroup of Γ not contained in Δ and set $\Delta' = \Delta \cap \Gamma'$. Suppose also that $\chi : \Gamma \to R^\times$ is a homomorphism. Let $r' : \Gamma' \to \mathcal{G}_n(R)$ be a homomorphism with $\nu \circ r' = \chi|_{\Gamma'}$ and $\Delta' = (r')^{-1}(GL_n(R) \times R^\times)$. Suppose $\gamma_0 \in \Gamma' - \Delta'$ and that r' corresponds to a triple $(\rho', \chi|_{\Gamma'}, \langle\ ,\ \rangle')$ as in lemma 2.1.1. We define

$$\mathrm{Ind}_{\Gamma',\Delta'}^{\Gamma,\Delta,\chi} r' : \Gamma \to \mathcal{G}_{n[\Gamma:\Gamma']}(R)$$

to be the homomorphism corresponding to the triple $(\rho, \chi, \langle\ ,\ \rangle)$ where ρ acts by right translation on the R-module of functions $f : \Delta \to R^n$ such that

$$f(\delta'\delta) = \rho'(\delta')f(\delta)$$

for all $\delta' \in \Delta'$ and $\delta \in \Delta$. We set

$$\langle f, f' \rangle = \sum_{\delta \in \Delta' \backslash \Delta} \chi(\delta)^{-1} \langle f(\delta), f'(\gamma_0 \delta \gamma_0^{-1}) \rangle'.$$

This construction is independent of the choice of γ_0 and we have

$$\nu \circ (\mathrm{Ind}\,_{\Gamma',\Delta'}^{\Gamma,\Delta,\chi} r') = \chi.$$

We will sometimes write $\mathrm{Ind}\,_{\Gamma'}^{\Gamma,\chi}$ for $\mathrm{Ind}\,_{\Gamma',\Delta'}^{\Gamma,\Delta,\chi}$, although it depends essentially on Δ as well as Γ', Γ and χ.

Now we consider the case that R is a field.

Lemma 2.1.3 *Suppose that k is a field of characteristic $\neq 2$ and that $r : \Gamma \to \mathcal{G}_n(k)$ such that $\Delta = r^{-1}(GL_n \times GL_1)(k)$. If $c \in \Gamma - \Delta$ and $c^2 = 1$, then*

$$\dim_k \mathfrak{g}_n^{c=\eta} = n(n + \eta(\nu \circ r)(c))/2$$

for $\eta = 1$ or -1.

Proof: We have $r(c) = (A, -(\nu \circ r)(c))\jmath$ where ${}^tA = -(\nu \circ r)(c)A$. Then

$$\mathfrak{g}_n^{c=\eta} = \{g \in M_n(k) : gA - \eta(\nu \circ r)(c)^t(gA) = 0\}.$$

The lemma follows. $\square$

Lemma 2.1.4 *Suppose that k is a field, that $\gamma_0 \in \Gamma - \Delta$, that $\chi : \Gamma \to k^\times$ is a homomorphism and that*

$$\rho : \Delta \longrightarrow GL_n(k)$$

is absolutely irreducible and satisfies $\chi\rho^\vee \cong \rho^{\gamma_0}$. Then there exists a homomorphism

$$r : \Gamma \longrightarrow \mathcal{G}_n(k)$$

such that $r|_\Delta = (\rho, \chi|_\Delta)$ and $r(\gamma_0) \in \mathcal{G}_n(k) - GL_n(k)$.

If $\alpha \in k^\times$ define

$$r_\alpha : \Gamma \longrightarrow \mathcal{G}_n(k)$$

by $r_\alpha|_\Delta = \rho$ and, if $\gamma \in \Gamma - \Delta$ and $r(\gamma) = (A, \mu)\jmath$, then

$$r_\alpha(\gamma) = (\alpha A, \mu)\jmath.$$

This sets up a bijection between $GL_n(k)$-conjugacy classes of extensions of ρ to $\Gamma \to \mathcal{G}_n(k)$ and $k^\times/(k^\times)^2$.

Note that $\nu \circ r_\alpha = \nu \circ r$. Also note that, if k is algebraically closed then r is unique up to $GL_n(k)$-conjugacy.

If Γ and R have topologies and ρ is continuous then so is r.

Proof: There exists a perfect pairing

$$\langle\ ,\ \rangle : k^n \times k^n \longrightarrow k$$

such that $\chi(\delta)\langle \rho(\delta)^{-1}x, y\rangle = \langle x, \rho(\gamma_0\delta\gamma_0^{-1})y\rangle$ for all $\delta \in \Delta$ and all $x, y \in k^n$. The absolute irreducibility of ρ implies that $\langle\ ,\ \rangle$ is unique up to $k^\times$-multiples. If we set

$$\langle x, y\rangle' = \langle y, \rho(\gamma_0^2)x\rangle$$

then $\chi(\delta)\langle \rho(\delta)^{-1}x, y\rangle' = \langle x, \rho(\gamma_0\delta\gamma_0^{-1})y\rangle'$ for all $\delta \in \Delta$ and all $x, y \in k^n$. Thus

$$\langle\ ,\ \rangle' = \varepsilon\langle\ ,\ \rangle$$

for some $\varepsilon \in k^\times$. As

$$\langle\ ,\ \rangle'' = \chi(\gamma_0^2)\langle\ ,\ \rangle$$

we see that $\varepsilon^2 = \chi(\gamma_0)^2$. The first assertion now follows from lemma 2.1.1. For the second assertion note that conjugation by $\alpha \in k^\times \subset GL_n(k)$ leaves ρ unchanged and replaces $\langle\ ,\ \rangle$ by $\alpha^2\langle\ ,\ \rangle$. $\square$

Lemma 2.1.5 *Suppose that Γ is profinite and that*

$$r : \Gamma \longrightarrow \mathcal{G}_n(\mathbf{Q}_l^{ac})$$

is a continuous representation with $\Delta = r^{-1}(GL_n \times GL_1)(\mathbf{Q}_l^{ac})$. Then there exists a finite extension $K/\mathbf{Q}_l$ and a continuous representation

$$r' : \Gamma \longrightarrow \mathcal{G}_n(\mathcal{O}_K)$$

which is $GL_n(\mathbf{Q}_l^{ac})$-conjugate to r.

Proof: By the Baire category theorem, the image $r(\Gamma)$ is a Baire space. It is also a countable union of closed subgroups:

$$r(\Gamma) = \bigcup_K (r(\Gamma) \cap \mathcal{G}_n(K))$$

where K runs over finite extensions of $\mathbf{Q}_l$ in $\overline{\mathbf{Q}}_l$. Thus one of the groups $r(\Gamma) \cap \mathcal{G}_n(K)$ contains a non-empty open subset of $r(\Gamma)$, and hence is of finite index in $r(\Gamma)$. It follows that $r(\Gamma) \subset \mathcal{G}_n(K)$ for some (possibly larger) finite extension $K/\mathbf{Q}_l$. A standard argument using the compactness of Δ

shows that there is a Δ-invariant $\mathcal{O}_K$-lattice $\Lambda \subset K^n$. (Choose any lattice and add it to all its translates by elements of Δ.) We may further suppose that the $\langle\ ,\ \rangle$-dual lattice Λ^* contains Λ. (If not replace Λ by a suitable scalar multiple.) Choose a maximal Δ-invariant $\mathcal{O}_K$-lattice $\Lambda^* \supset M \supset \Lambda$ such that $M^* \supset M$, and replace Λ by M. Then if $\Lambda^* \supset N \supset \Lambda$ is any Δ-invariant $\mathcal{O}_K$-lattice with N/Λ simple, we must have $N^* \cap N = \Lambda$. We conclude that Λ^*/Λ must be a direct sum of simple $\mathcal{O}_K[\Delta]$-modules. Replacing K by a ramified quadratic extension and repeating this procedure we get a Δ-invariant $\mathcal{O}_K$-lattice Λ with $\Lambda^* = \Lambda$. The lemma now follows from lemma 2.1.1. $\Box$

Deformation theory works well for absolutely irreducible representations $\Gamma \to GL_n(k)$. In the case of homomorphisms $r : \Gamma \to \mathcal{G}_n(k)$ with $\Delta = r^{-1}(GL_n \times GL_1)(k)$, it works well if $r|_\Delta$ is absolutely irreducible. However it seems to work equally well in slightly greater generality. To express this we make the following definition. For our applications to modularity lifting theorems and to the Sato-Tate conjecture the case $r|_\Delta$ absolutely irreducible will suffice, so the reader who is only interested in these applications can simply read "$r|_\Delta$ absolutely irreducible" for "Schur".

Definition 2.1.6 *Suppose that k is a field and $r : \Gamma \to \mathcal{G}_n(k)$ is a homomorphism with $\Delta = r^{-1}(GL_n \times GL_1)(k)$. Let $\gamma_0 \in \Gamma - \Delta$. We will call r* Schur *if all irreducible Δ-subquotients of k^n are absolutely irreducible and if for all Δ-invariant subspaces $k^n \supset W_1 \supset W_2$ with k^n/W_1 and W_2 irreducible, we have*

$$W_2^\vee(\nu \circ r) \not\cong (k^n/W_1)^{\gamma_0}.$$

This is independent of the choice of γ_0.

Note that if $r|_\Delta$ is absolutely irreducible then r is certainly Schur. Also note that if k'/k is a field extension then $r : \Gamma \to \mathcal{G}_n(k)$ is Schur if and only if $r : \Gamma \to \mathcal{G}_n(k')$ is.

Lemma 2.1.7 *Suppose that k is a field and $r : \Gamma \to \mathcal{G}_n(k)$ is a homomorphism with $\Delta = r^{-1}(GL_n \times GL_1)(k)$. If r is Schur then the following assertions hold.*

1. *$r|_\Delta$ is semisimple.*
2. *If $r' : \Gamma \to \mathcal{G}_n(k)$ is another representation with $\Delta = (r')^{-1}(GL_n \times GL_1)(k)$ and $\operatorname{tr} r|_\Delta = \operatorname{tr} r'|_\Delta$, then r' is $GL_n(k^{ac})$-conjugate to r.*
3. *If k does not have characteristic 2 then $\mathfrak{g}_n^\Gamma = (0)$.*

Proof: We may suppose that k is algebraically closed.

Choose $\gamma_0 \in \Gamma - \Delta$. Suppose that r corresponds to $(r|_\Delta, \mu, \langle\ ,\ \rangle)$ as in lemma 2.1.1, and let $V \subset k^n$ be an irreducible Δ-submodule. Then $(k^n/V^\perp)^{\gamma_0}$

is isomorphic to $V^\vee(\nu \circ r)$, and so we can not have $V \subset V^\perp$. Thus $k^n \cong V \oplus V^\perp$ as Δ-modules. Arguing recursively we see that we have a decomposition

$$k^n \cong V_1 \oplus ... \oplus V_r$$

and

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_1 \perp ... \perp \langle\ ,\ \rangle_r,$$

where each V_i is an irreducible $k[\Delta]$-module and each $\langle\ ,\ \rangle_i$ is a perfect pairing on V_i. The first part of the lemma follows. Note also that for $i \neq j$ we have $V_i \not\cong V_j$ as $k[\Delta]$-modules and $V_i^{\gamma_0} \cong V_i^\vee(\nu \circ r)$.

Note that if ρ and τ are representations $\Delta \to GL_n(k)$ with ρ semisimple and multiplicity free and with $\mathrm{tr}\,\rho = \mathrm{tr}\,\tau$, then the semisimplification of τ is equivalent to ρ. Thus $r'|_\Delta$ has the same Jordan-Holder factors as $r|_\Delta$ (with multiplicity). Thus r' satisfies the same hypothesis as r and so by part one $r'|_\Delta$ is also semisimple. Hence $r'|_\Delta \cong r|_\Delta$, and we may suppose that in fact $r'|_\Delta = r|_\Delta$. Then corresponding to our decomposition

$$k^n \cong V_1 \oplus ... \oplus V_r$$

we see that r corresponds to

$$(r|_\Delta, \mu, \langle\ ,\ \rangle_1 \perp ... \perp \langle\ ,\ \rangle_r)$$

while r' corresponds to

$$(r|_\Delta, \mu, \mu_1\langle\ ,\ \rangle_1 \perp ... \perp \mu_r\langle\ ,\ \rangle_r)$$

for some $\mu_i \in k^\times$. Conjugation by the element of $GL_n(k)$ which acts on V_i by $\sqrt{\mu_i}$ takes r to r'.

For the third part note that

$$\mathfrak{g}_n^\Delta = \mathrm{End}_{k[\Delta]}(V_1) \oplus ... \oplus \mathrm{End}_{k[\Delta]}(V_r) = k^r.$$

Then γ_0 sends $(\alpha_1, ..., \alpha_r)$ to $(-\alpha_1^{*_1}, ..., -\alpha_r^{*_r}) = (-\alpha_1, ..., -\alpha_r)$, where $*_i$ denotes the adjoint with respect to $\langle\ ,\ \rangle_i$. Thus $\mathfrak{g}_n^\Gamma = (0)$. $\square$

We now turn to the case that R is a noetherian complete local ring. We first recall the well known case of homomorphisms to $GL_n(R)$, before studying homomorphisms to $\mathcal{G}_n(R)$.

Lemma 2.1.8 *Let R be a noetherian complete local ring. Let Δ be a profinite group and $\rho : \Delta \longrightarrow GL_n(R)$ a continuous representation. Suppose that $\rho \bmod \mathfrak{m}_R$ is absolutely irreducible. Then the centraliser in $GL_n(R)$ of the image of ρ is $R^\times$.*

Proof: It suffices to consider the case that R is Artinian. We can then argue by induction on the length of R. The case R is a field is well known. So suppose that I is a non-zero ideal of R with $\mathfrak{m}_R I = (0)$. If z is an element of $Z_{GL_n(R)}(\mathrm{Im}\,\rho)$ then we see by the inductive hypothesis that $z \in R^\times(1 + M_n(I))$. With out loss of generality we can suppose $z = 1 + y \in 1 + M_n(I)$. Thus $y \in (\mathrm{ad}\,(\rho \bmod \mathfrak{m}_R))^{\Delta} \otimes_{R/\mathfrak{m}_R} I = I$, and the lemma is proved. $\Box$

Lemma 2.1.9 *Let $R \supset S$ be noetherian complete local rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field. Let Δ be a profinite group and let $\rho, \rho' : \Delta \longrightarrow GL_n(S)$ be continuous representations. Suppose that for all ideals $I \subset J$ of R we have*

$$Z_{1+M_n(\mathfrak{m}_R/I)}(\mathrm{Im}\,(\rho \bmod I)) \twoheadrightarrow Z_{1+M_n(\mathfrak{m}_R/J)}(\mathrm{Im}\,(\rho \bmod J)).$$

(This will be satisfied if, for instance, $\rho \bmod \mathfrak{m}_S$ is absolutely irreducible.) If ρ and ρ' are conjugate in $GL_n(R)$ then they are conjugate in $GL_n(S)$.

Proof: It suffices to consider the case that R is Artinian (because $S = \lim_{\leftarrow} S/I \cap S$ as I runs over open ideals of R). Again we argue by induction on the length of R. If R is a field there is nothing to do. So suppose that I is an ideal of R and $\mathfrak{m}_R I = (0)$. By the inductive hypothesis we may suppose that $\rho \bmod I \cap S = \rho' \bmod I \cap S$. Thus $\rho' = (1 + \phi)\rho$ for some cocycle $\phi \in Z^1(\Delta, \mathrm{ad}\,(\rho \bmod \mathfrak{m}_S)) \otimes (I \cap S)$. As ρ and ρ' are conjugate in R, our assumption (on surjections of centralisers) tells us that they are conjugate by an element of $1 + M_n(I)$. Hence $[\phi] = 0$ in $H^1(\Delta, \mathrm{ad}\,(\rho \bmod \mathfrak{m}_S)) \otimes I$. Thus $[\phi] = 0$ in $H^1(\Delta, \mathrm{ad}\,(\rho \bmod \mathfrak{m}_S)) \otimes (I \cap S)$, so that ρ and ρ' are conjugate by an element of $1 + M_n(I \cap S)$. $\Box$

Lemma 2.1.10 (Carayol) *Let $R \supset S$ be noetherian complete local rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field. Let Δ be a profinite group and $\rho : \Delta \longrightarrow GL_n(R)$ a continuous representation. Suppose that $\rho \bmod \mathfrak{m}_R$ is absolutely irreducible and that $\mathrm{tr}\,\rho(\Delta) \subset S$. If I is an ideal of R such that $\rho \bmod I$ has image in $S/I \cap S$, then there is a $1_n + M_n(I)$-conjugate ρ' of ρ such that the image of ρ' is contained in $GL_n(S)$. In particular there is always a $1_n + M_n(\mathfrak{m}_R)$-conjugate ρ' of ρ such that the image of ρ' is contained in $GL_n(S)$.*

Proof: A simple recursion allows one to reduce to the case that $\mathfrak{m}_R I = (0)$ and $\dim_{R/\mathfrak{m}_R} I = 1$. Replacing R by the set of elements in R which are congruent mod I to an element of S we may further assume that $S/I \cap S \xrightarrow{\sim} R/I$. If $I \subset S$ then $R = S$ and there is nothing to prove. Otherwise $R = S \oplus I$ with multiplication

$$(s, i)(s', i') = (ss', s'i + si').$$

Note that $\mathfrak{m}_S R = \mathfrak{m}_S$, that $R/\mathfrak{m}_S \cong (S/\mathfrak{m}_S)[\epsilon]/(\epsilon^2)$ and that an element $r \in R$ lies in S if and only if $r \bmod \mathfrak{m}_S$ lies in $S/\mathfrak{m}_S$. Suppose we know the result for $S/\mathfrak{m}_S \subset R/\mathfrak{m}_S$. Then we can find $A \in M_n(I)$ such that

$$(1_n - A)\rho(1_n + A) \bmod \mathfrak{m}_S$$

is valued in $GL_n(S/\mathfrak{m}_S)$ so that

$$(1_n - A)\rho(1_n + A)$$

is valued in S. Hence the result would follow for $S \subset R$.

Thus we are reduced to the case $S = k$ is a field, $R = k[\epsilon]/(\epsilon^2)$ and $I = (\epsilon)$. Extend ρ to a homomorphism

$$\rho : k[\Delta] \longrightarrow M_n(R).$$

Note that $\rho \bmod \epsilon$ is surjective onto $M_n(k)$, and write J for the kernel of $\rho \bmod \epsilon$. If $\delta \in k[\Delta]$ and $\gamma \in J$ then

$$\operatorname{tr} \rho(\delta)(\rho(\gamma)/\epsilon) = 0.$$

Thus

$$\operatorname{tr} M_n(k)(\rho(\gamma)/\epsilon) = (0)$$

and $\rho(\gamma) = 0$. We deduce that ρ factors through $(\rho \bmod \epsilon) : k[\Delta] \to M_n(k)$, i.e.

$$\rho(\delta) = (\rho \bmod \epsilon)(\delta) + \phi((\rho \bmod \epsilon)(\delta))\epsilon$$

where

$$\phi : M_n(k) \longrightarrow M_n(k)$$

is a k-linear map satisfying

$$\phi(ab) = a\phi(b) + \phi(a)b.$$

There is an element $A \in M_n(k)$ such that

$$\phi(b) = Ab - bA.$$

(See for instance lemma 1 of [Ca]. Alternatively it is not hard to check that

$$A = \sum_{j=1}^{n} \phi(e_{j,1})e_{1,j}$$

will work, where $e_{i,j}$ denotes the matrix which has a 1 in the intersection of the i^{th} row and j^{th} column, and zeros elsewhere.) Then

$$(1_n - A\epsilon)\rho(1 + A\epsilon) = (\rho \bmod \epsilon)$$

is valued in $M_n(k)$, and the lemma follows. □

Finally in this section we turn to analogous results for homomorphisms into $\mathcal{G}_n(R)$.

Lemma 2.1.11 *Let R be a complete local noetherian ring with maximal ideal $\mathfrak{m}_R$ and residue field $k = R/\mathfrak{m}_R$ of characteristic $l > 2$. Let Γ be a group and let $r : \Gamma \to \mathcal{G}_n(R)$ be a homomorphism such that $\Delta = r^{-1}(GL_n \times GL_1)(R)$ has index 2 in Γ. Suppose moreover that $r \bmod \mathfrak{m}_R$ is Schur. (Which is true if, for instance, $r|_\Delta \bmod \mathfrak{m}_R$ is absolutely irreducible.) Then the centraliser of r in $1 + M_n(\mathfrak{m}_R)$ is $\{1\}$.*

Proof: This lemma is easily reduced to the case that R is Artinian. In this case we argue by induction on the length of R, the case of length 1 (i.e. $R = k$) being immediate. In general we may choose an ideal I of R such that I has length 1. By the inductive hypothesis any element of the centraliser in $1+M_n(\mathfrak{m}_R)$ of the image of r lies in $1+M_n(I)$. It follows from lemma 2.1.7 that this centraliser is $\{1\}$. $\Box$

Lemma 2.1.12 *Suppose that $R \supset S$ are complete local noetherian rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field k of characteristic $l > 2$. Suppose that Γ is a profinite group and that $r : \Gamma \to \mathcal{G}_n(R)$ is a continuous representation with $\Delta = r^{-1}(GL_n \times GL_1)(R)$. Suppose moreover that $r|_\Delta \bmod \mathfrak{m}_R$ is absolutely irreducible and that $\operatorname{tr} r(\Delta) \subset S$. Then r is $GL_n(R)$-conjugate to a homomorphism $r' : \Gamma \to \mathcal{G}_n(S)$.*

Proof: By lemma 2.1.10 we may suppose that $r(\Delta) \subset (GL_n \times GL_1)(S)$. Choose $\gamma_0 \in \Gamma - \Delta$ and suppose $r(\gamma_0) = (A, -\mu)\jmath$ with $A \in GL_n(R)$. Then

$$r|_\Delta^{\gamma_0} = Ar|_\Delta^\vee(\nu \circ r)A^{-1}.$$

It follows from lemma 2.1.9 that we can find $B \in GL_n(S)$-conjugate with

$$r|_\Delta^{\gamma_0} = Br|_\Delta^\vee(\nu \circ r)B^{-1}.$$

It follows from lemma 2.1.8 that $A = \alpha B$ for some $\alpha \in R^\times$. As R and S have the same residue field we may choose B so that $\alpha \in 1 + \mathfrak{m}_R$. Then $\alpha = \beta^2$ for some $\beta \in R^\times$ and

$$(\beta 1_n, 1)r(\gamma_0)(\beta 1_n, 1)^{-1} \in \mathcal{G}_n(S).$$

Thus

$$(\beta 1_n, 1)r(\beta 1_n, 1)^{-1}$$

is valued in $\mathcal{G}_n(S)$, as desired. $\Box$

We remark that *this* lemma does not remain true of the hypothesis that $r|_\Delta \bmod \mathfrak{m}R$ is absolutely irreducible is weakened to $r \bmod \mathfrak{m}_R$ is Schur.

2.2. *Deformation theory.* — In this section we will discuss the deformation theory of homomorphisms into $\mathcal{G}_n$. This closely mirrors Mazur's deformation theory of representations of Galois groups, but the section gives us an opportunity both to generalise the results to $\mathcal{G}_n$ and to set things up in a way that will be convenient in the sequel. At the referees' suggestion we include a discussion of Kisin's framed deformations which originally appeared in [Tay].

Let l be an odd prime. Let k denote an algebraic extension of the finite field with l elements, let $\mathcal{O}$ denote the ring of integers of a finite totally ramified extension K of the fraction field of the Witt vectors $W(k)$, let λ denote the maximal ideal of $\mathcal{O}$, let $\mathcal{C}^f_{\mathcal{O}}$ denote the category of Artinian local $\mathcal{O}$-algebras for which the structure map $\mathcal{O} \to R$ induces an isomorphism on residue fields, and let $\mathcal{C}_{\mathcal{O}}$ denote the full subcategory of the category of topological $\mathcal{O}$-algebras whose objects are inverse limits of objects of $\mathcal{C}^f_{\mathcal{O}}$. The morphisms in $\mathcal{C}^f_{\mathcal{O}}$ and $\mathcal{C}_{\mathcal{O}}$ are continuous homomorphisms of $\mathcal{O}$-algebras which induce isomorphisms on the residue fields. Also fix a profinite group Γ together with a closed subgroup $\Delta \subset \Gamma$ of index 2. Also fix a continuous homomorphism

$$\overline{r} : \Gamma \longrightarrow \mathcal{G}_n(k)$$

and a homomorphism $\chi : \Gamma \to \mathcal{O}^\times$, such that $\Delta = \overline{r}^{-1}(GL_n \times GL_1)(k)$ and $\nu \circ \overline{r} = \chi$. Let S be a finite index set. For $q \in S$ let Δ_q be a topologically finitely generated profinite group provided with a continuous homomorphism $\Delta_q \to \Delta$. In applications Γ will be a global Galois group and Δ_q will be a local Galois group. We will sometimes write $\overline{r}|_{\Delta_q}$ for the composite

$$\Delta_q \longrightarrow \Delta \xrightarrow{\overline{r}} \mathcal{G}^0_n(k) \twoheadrightarrow GL_n(k).$$

We will want to distinguish between 'liftings' of representations and conjugacy classes of liftings, which we will refer to as 'deformations'.

Definition 2.2.1 *By a* lifting *of $\overline{r}$ (resp. $\overline{r}|_{\Delta_q}$) to an object R of $\mathcal{C}_{\mathcal{O}}$ we shall mean a continuous homomorphism $r : \Gamma \to \mathcal{G}_n(R)$ (resp. $r : \Delta_q \to GL_n(R)$) with $r \bmod \mathfrak{m}_R = \overline{r}$ (resp. $= \overline{r}|_{\Delta_q}$) and (in the former case) $\nu \circ r = \chi$. We will call two liftings* equivalent *if they are conjugate by an element of $1 + M_n(\mathfrak{m}_R) \subset GL_n(R)$. By a* deformation *of $\overline{r}$ (resp. $\overline{r}|_{\Delta_q}$) we shall mean an equivalence class of liftings.*

*Let $T \subset S$. By a T-*framed lifting *of $\overline{r}$ to R we mean a tuple $(r; \alpha_q)_{q\in T}$ where r is a lifting of $\overline{r}$ and $\alpha_q \in 1 + M_n(\mathfrak{m}_R)$. We call two framed liftings $(r; \alpha_q)_{q\in T}$ and $(r'; \alpha'_q)_{q\in T}$ are called* equivalent *if there is an element $\beta \in 1_n + M_n(\mathfrak{m}_R)$ with $r' = \beta r \beta^{-1}$ and $\alpha'_q = \beta\alpha_q$. By a T-*framed deformation *of $\overline{r}$ we shall mean an equivalence class of framed liftings. If $T = S$ we shall simply refer to* framed liftings *and* framed deformations.

Note that we can associate to a T-framed deformation $[(r;\alpha_q)_{q\in T}]$ of $\overline{r}$ both a deformation $[r]$ of $\overline{r}$ and, for $q\in T$, a lifting $\alpha_q^{-1}r|_{\Delta_q}\alpha_q$ of $\overline{r}|_{\Delta_q}$. (Here we define $r|_{\Delta_q}$ in the same manner we defined $\overline{r}|_{\Delta_q}$ above.)

For $q\in S$ there is a universal lifting (*not* deformation)

$$r_q^{\mathrm{univ}}:\Delta_q\longrightarrow GL_n(R_q^{\mathrm{loc}})$$

of $\overline{r}|_{\Delta_q}$ over an object R_q^{loc} of $\mathcal{C}_{\mathcal{O}}$. As Δ_q is topologically finitely generated, R_q^{loc} is noetherian. (A lifting is determined by the images of a set of topological generators for Δ_q.) Note that R_q^{loc} has a natural (left) action of $1_n+M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$. (An element $g\in 1_n+M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ acts via the map $R_q^{\mathrm{loc}}\to R_q^{\mathrm{loc}}$ under which r_q^{univ} pulls back to $gr_q^{\mathrm{univ}}g^{-1}$.) There are natural isomorphisms

$$\mathrm{Hom}_{\,k}(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}},\lambda),k)\cong \mathrm{Hom}_{\,\mathcal{C}_{\mathcal{O}}}(R_q^{\mathrm{loc}},k[\epsilon]/(\epsilon^2))\cong Z^1(\Delta_q,\mathrm{ad}\,\overline{r}).$$

The first is standard. Under the second a cocycle $\phi\in Z^1(\Delta_q,\mathrm{ad}\,\overline{r})$ corresponds to the homomorphism arising from the lifting

$$(1+\phi\epsilon)\overline{r}|_{\Delta_q}$$

of $\overline{r}|_{\Delta_q}$. The action of $M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}},\lambda))$ on $R_q^{\mathrm{loc}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}},\lambda)$ gives an action on $Z^1(\Delta_q,\mathrm{ad}\,\overline{r})$ which can be described as follows. Recall that we have an exact sequence

$$(0)\to H^0(\Delta_q,\mathrm{ad}\,\overline{r})\to \mathrm{ad}\,\overline{r}\to Z^1(\Delta_q,\mathrm{ad}\,\overline{r})\to H^1(\Delta_q,\mathrm{ad}\,\overline{r})\to(0).$$

If $\psi\in \mathrm{Hom}_{\,k}(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}},\lambda),k)$ corresponds to $z\in Z^1(\Delta_q,\mathrm{ad}\,\overline{r})$, then $B\in M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}},\lambda))$ takes z to z plus the image of $\psi(B)\in\mathrm{ad}\,\overline{r}$. In particular there is a bijection between $M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}},\lambda))$ invariant subspaces of $Z^1(\Delta_q,\mathrm{ad}\,\overline{r})$ and subspaces of $H^1(\Delta_q,\mathrm{ad}\,\overline{r})$.

Let R be an object of $\mathcal{C}_{\mathcal{O}}$ and I be a closed ideal of R with $\mathfrak{m}_RI=(0)$. Suppose that r_1 and r_2 are two liftings of $\overline{r}|_{\Delta_q}$ with the same reduction mod I. Then

$$\gamma\longmapsto r_2(\gamma)r_1(\gamma)^{-1}-1$$

defines an element of $H^1(\Delta_q,\mathrm{ad}\,\overline{r})\otimes_k I$ which we shall denote $[r_2-r_1]$. In fact this sets up a bijection between $H^1(\Delta_q,\mathrm{ad}\,\overline{r})\otimes_k I$ and $(1+M_n(I))$-conjugacy classes of lifts which agree with r_1 modulo I. Now suppose that r is a lift of $\overline{r}|_{\Delta_q}$, to R/I. For each $\gamma\in\Delta_q$ choose a lifting $\widetilde{r(\gamma)}$ to $GL_n(R)$ of $r(\gamma)$. Then

$$(\gamma,\delta)\longmapsto \widetilde{r(\gamma\delta)}\,\widetilde{r(\delta)}^{-1}\,\widetilde{r(\gamma)}^{-1}$$

defines a class $\mathrm{obs}_{R,I}(r) \in H^2(\Delta_q, \mathrm{ad}\,\overline{r}) \otimes_k I$ which is independent of the choices made and vanishes if and only if r lifts to R.

Now suppose that r_q is a lifting of $\overline{r}|_{\Delta_q}$ to $\mathcal{O}$ corresponding to a homomorphism $\alpha : R_q^{\mathrm{loc}} \to \mathcal{O}$. There is also a natural identification

$$\mathrm{Hom}_{\mathcal{O}}(\ker\alpha/(\ker\alpha)^2, K/\mathcal{O}) \cong Z^1(\Delta_q, \mathrm{ad}\, r_q \otimes K/\mathcal{O}).$$

This may be described as follows. Consider the topological $\mathcal{O}$-algebra $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ where $\epsilon^2 = 0$. Although $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ is not an object of $\mathcal{C}_{\mathcal{O}}$, it still makes sense to talk about liftings of r_q to $\mathcal{O} \oplus K/\mathcal{O}\epsilon$. One can then check that such liftings are parametrised by $Z^1(\Delta_q, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$. (Any such lifting arises from a lifting to some $\mathcal{O} \oplus \lambda^{-N}\mathcal{O}/\mathcal{O}\epsilon$.) On the other hand such liftings correspond to homomorphisms $R_q^{\mathrm{loc}} \to \mathcal{O} \oplus K/\mathcal{O}\epsilon$ lifting α and such liftings correspond to $\mathrm{Hom}_{\mathcal{O}}(\ker\alpha/(\ker\alpha)^2, K/\mathcal{O})$.

Definition 2.2.2 *If $q \in S$ then by a* local deformation problem *at q we mean a collection $\mathcal{D}_q$ of liftings of $\overline{r}|_{\Delta_q}$ to objects of $\mathcal{C}_{\mathcal{O}}$ satisfying the following conditions.*

1. *$(k, \overline{r}|_{\Delta_q}) \in \mathcal{D}_q$.*
2. *If $(R, r) \in \mathcal{D}_q$ and if $f : R \to S$ is a morphism in $\mathcal{C}_{\mathcal{O}}$ then $(S, f \circ r) \in \mathcal{D}_q$.*
3. *Suppose that (R_1, r_1) and $(R_2, r_2) \in \mathcal{D}_q$, that I_1 (resp. I_2) is a closed ideal of R_1 (resp. R_2) and that $f : R_1/I_1 \xrightarrow{\sim} R_2/I_2$ is an isomorphism in $\mathcal{C}_{\mathcal{O}}$ such that $f(r_1 \bmod I_1) = (r_2, \mathrm{mod} I_2)$. Let R_3 denote the subring of $R_1 \oplus R_2$ consisting of pairs with the same image in $R_1/I_1 \xrightarrow{\sim} R_2/I_2$. Then $(R_3, r_1 \oplus r_2) \in \mathcal{D}_q$.*
4. *If (R_j, r_j) is an inverse system of elements of $\mathcal{D}_q$ then*
$$(\lim_{\leftarrow} R_j, \lim_{\leftarrow} r_j) \in \mathcal{D}_q.$$
5. *$\mathcal{D}_q$ is closed under equivalence.*
6. *If $R \hookrightarrow S$ is an injective morphism in $\mathcal{C}_{\mathcal{O}}$ and if $r : \Delta_q \to GL_n(R)$ is a lifting of $\overline{r}|_{\Delta_q}$ such that $(S, r) \in \mathcal{D}_q$ then $(R, r) \in \mathcal{D}_q$.*

(Compare with section 23 of [Maz].)

Lemma 2.2.3 *If $\mathcal{I}$ is a $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant ideal of R_q^{loc} then the collection of all liftings r over rings R such that the kernel of the induced map $R_q^{\mathrm{loc}} \to R$ contains $\mathcal{I}$ is a local deformation problem. Moreover every local deformation problem $\mathcal{D}_q$ arises in this way from some $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant ideal $\mathcal{I}_q$ of R_q^{loc}.*

Proof: The first assertion is clear. Consider the second assertion. Let $\mathfrak{I}$ denote the set of ideals $\mathcal{I}$ of R_q^{loc} such that $(R_q^{\mathrm{loc}}/\mathcal{I}, r_q^{\mathrm{univ}}) \in \mathcal{D}_q$. The second and sixth conditions on $\mathcal{D}_q$ tell us that a lifting (R, r) of $\overline{r}|_{\Delta_q}$ lies in $\mathcal{D}_q$ if and only if the kernel of the corresponding map $R_q^{\mathrm{loc}} \to R$ lies in $\mathfrak{I}$. The first condition on $\mathcal{D}_q$ tells us that $\mathfrak{I}$ is non-empty, the third condition tells us it is closed under finite intersections and the fourth condition tells us that it is closed under arbitrary nested intersections. Thus $\mathfrak{I}$ contains a minimal element $\mathcal{I}_q$ which is contained in all other elements of $\mathfrak{I}$. The second condition on $\mathcal{D}_q$ tells us that any ideal of R_q^{loc} containing $\mathcal{I}_q$ lies in $\mathfrak{I}_q$ and the second assertion follows. $\square$

Definition 2.2.4 *Suppose $\mathcal{D}_q$ is a local deformation problem corresponding to an ideal $\mathcal{I}_q$ of R_q^{loc}. We will write $L_q = L_q(\mathcal{D}_q)$ for the image in $H^1(\Delta_q, \mathrm{ad}\,\overline{r})$ of the annihilator L_q^1 in $Z^1(\Delta_q, \mathrm{ad}\,\overline{r})$ of $\mathcal{I}_q/(\mathcal{I}_q \cap (\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda)) \subset \mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda)$ under the isomorphism*

$$\mathrm{Hom}\,_k(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda), k) \cong Z^1(\Delta_q, \mathrm{ad}\,\overline{r}).$$

Because $\mathcal{I}_q$ is $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant we see that L_q^1 is the preimage of L_q in $Z^1(\Delta_q, \mathrm{ad}\,\overline{r})$.

We remark that

$$\mathrm{Hom}\,_k(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \mathcal{I}_q, \lambda), k) \cong L_q^1$$

and the exact sequence in the paragraph after definition 2.2.1 shows that

$$\dim_k L_q^1 = n^2 + \dim_k L_q - \dim_k H^0(\Delta_q, \mathrm{ad}\,\overline{r}).$$

Lemma 2.2.5 *Keep the above notation and assumptions. Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and I is a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that r_1 and r_2 are two liftings of $\overline{r}|_{\Delta_q}$ with the same reduction* $\mathrm{mod}\, I$. *Suppose finally that r_1 is in $\mathcal{D}_q$. Then r_2 is in $\mathcal{D}_q$ if and only if $[r_2 - r_1] \in L_q$.*

Proof: Suppose that r_j corresponds to $\alpha_j : R_q^{\mathrm{loc}} \to R$. Then $\alpha_2 = \alpha_1 + \beta$ where

$$\beta : R_q^{\mathrm{loc}} \longrightarrow I$$

satisfies

- $\beta(x + y) = \beta(x) + \beta(y)$;
- $\beta(xy) = \beta(x)\alpha_1(y) + \alpha_1(x)\beta(y)$;
- and $\beta|_{\mathcal{O}} = 0$.

Thus β is determined by $\beta|_{\mathfrak{m}_{R_q^{\mathrm{loc}}}}$ and β is trivial on $(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda)$. Hence β gives rise to and is determined by an $\mathcal{O}$-linear map:

$$\beta : \mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda) \longrightarrow I.$$

A straightforward calculation shows that

$$[r_2 - r_1] \in H^1(\Delta_q, \mathrm{ad}\, \overline{r})$$

is the image of

$$\beta \in \mathrm{Hom}\, (\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda), I) \cong Z^1(\Delta_q, \mathrm{ad}\, \overline{r}) \otimes_k I.$$

The homomorphism α_1 vanishes on $\mathcal{I}_q$. Thus we must show that β vanishes on $\mathcal{I}_q$ if and only if β maps to $L_q \otimes_k I$, i.e. if and only if

$$\beta \in \mathrm{Hom}\, (\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda, \mathcal{I}_q), k) \otimes_k I.$$

This is tautological. $\Box$

Again let r_q be a lift of $\overline{r}|_{\Delta_q}$ to $\mathcal{O}$ corresponding to a homomorphism $\alpha : R_q^{\mathrm{loc}} \to \mathcal{O}$. Suppose that r_q is in $\mathcal{D}_q$. We will call a lift of r_q to $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ *of type* $\mathcal{D}_q$ if it arises by extension of scalars from a lift to some $\mathcal{O} \oplus \lambda^{-N}\mathcal{O}/\mathcal{O}\epsilon$ which is in $\mathcal{D}_q$. Such liftings correspond to homomorphisms $R_q^{\mathrm{loc}}/\mathcal{I}_q \to \mathcal{O} \oplus K/\mathcal{O}\epsilon$ which lift α. Because $\mathcal{I}_q$ is $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant, the subspace of $Z^1(\Delta_q, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ corresponding to

$$\mathrm{Hom}_{\mathcal{O}}(\ker\alpha/((\ker\alpha)^2, \mathcal{I}_q), K/\mathcal{O}) \subset \mathrm{Hom}_{\mathcal{O}}(\ker\alpha/(\ker\alpha)^2, K/\mathcal{O})$$

is the inverse image of a sub-$\mathcal{O}$-module

$$L_q(r_q) \subset H^1(\Delta_q, \mathrm{ad}\, r_q \otimes K/\mathcal{O}).$$

Thus a lift of r_q to $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ is of type $\mathcal{D}_q$ if and only if its class in $Z^1(\Delta_q, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ maps to an element of $L_q(r_q)$.

Definition 2.2.6 *We will call* $\mathcal{D}_q$ liftable *if for each object* R *of* $\mathcal{C}_{\mathcal{O}}$*, for each ideal* I *of* R *with* $\mathfrak{m}_R I = (0)$ *and for each lifting* r *to* R/I *in* $\mathcal{D}_q$ *there is a lifting of* r *to* R*. This is equivalent to* $R_q^{\mathrm{loc}}/\mathcal{I}_q$ *being a power series ring over* $\mathcal{O}$*.*

We now turn to deformations of $\overline{r} : \Gamma \to \mathcal{G}_n(k)$.

Definition 2.2.7 *Let $\mathcal{S}$ be a collection of deformation problems $\mathcal{D}_q$ for each $q \in S$ and let $T \subset S$. We call a T-framed lifting $(R, r; \alpha_q)_{q\in T}$ of $\overline{r}$* of type $\mathcal{S}$ *if for all $q \in S$ the restriction $(R, r|_{\Delta_q}) \in \mathcal{D}_q$. For $q \in T$ this is equivalent to requiring $(R, \alpha_q^{-1} r|_{\Delta_q}\alpha_q)$ to lie in $\mathcal{D}_q$. If a T-framed lifting is of type $\mathcal{S}$, so is any equivalent T-framed lifting. We say that a T-framed deformation is of type $\mathcal{S}$ if some (or equivalently, every) element is of type $\mathcal{S}$. We let $\mathrm{Def}_{\mathcal{S}}^{\square_T}$ denote the functor from $\mathcal{C}_{\mathcal{O}}$ to sets which sends R to the set of T-framed deformations of $\overline{r}$ to R of type $\mathcal{S}$. If $T = S$ we shall refer simply to framed deformations and write $\mathrm{Def}_{\mathcal{S}}^{\square}$. If $T = \emptyset$ we shall refer simply to deformations and write $\mathrm{Def}_{\mathcal{S}}$.*

We need to introduce a variant of the cohomology groups $H^i(\Gamma, \mathrm{ad}\,\overline{r})$. More specifically we will denote by $H^i_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r})$ the cohomology of the complex

$$C^i_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) = C^i(\Gamma, \mathrm{ad}\,\overline{r}) \oplus \bigoplus_{q\in S} C^{i-1}(\Delta_q, \mathrm{ad}\,\overline{r})/M_q^{i-1},$$

where $M_q^i = (0)$ unless $q \in S - T$ and $i = 0$ in which case

$$M_q^0 = C^0(\Delta_q, \mathrm{ad}\,\overline{r}),$$

or $q \in S - T$ and $i = 1$, in which case $M_q^1 = L_q^1$ denotes the preimage of L_q in $C^1(\Delta_q, \mathrm{ad}\,\overline{r})$. The boundary map is

$$\begin{array}{rcl} C^i_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) & \longrightarrow & C^{i+1}_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) \\ (\phi, (\psi_q)) & \longmapsto & (\partial\phi, (\phi|_{\Delta_q} - \partial\psi_q)). \end{array}$$

If $T = \emptyset$ we will drop it from the notation. If $T = S$ we will drop the $\mathcal{S}$ from the notation.

We have a long exact sequence

$$\begin{array}{lllll} & & & (0) & \to \\ \to H^0_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) & \to H^0(\Gamma, \mathrm{ad}\,\overline{r}) & \to & \bigoplus_{q\in T} H^0(\Delta_q, \mathrm{ad}\,\overline{r}) & \to \\ \to H^1_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) & \to H^1(\Gamma, \mathrm{ad}\,\overline{r}) & \to & \left(\bigoplus_{q\in S-T} H^1(\Delta_q, \mathrm{ad}\,\overline{r})/L_q\right) & \\ & & & \oplus \left(\bigoplus_{q\in T} H^1(\Delta_q, \mathrm{ad}\,\overline{r})\right) & \to \\ \to H^2_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) & \to H^2(\Gamma, \mathrm{ad}\,\overline{r}) & \to & \bigoplus_{q\in S} H^2(\Delta_q, \mathrm{ad}\,\overline{r}) & \to \\ \to H^3_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) & \to H^3(\Gamma, \mathrm{ad}\,\overline{r}) & \to & \dots & \end{array}$$

Note that the dimensions of $H^i(\Gamma, \mathrm{ad}\,\overline{r})$ and $H^i_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r})$ are either both finite or both infinite.

(At least one of the authors thinks it is helpful to write that this is a special case of a 'cone construction'.)

Lemma 2.2.8 *Suppose that all the groups $H^i(\Gamma, \operatorname{ad}\overline{r})$ and $H^i(\Delta_q, \operatorname{ad}\overline{r})$ are finite and that they all vanish for i sufficiently large. Set*

$$\chi(\Gamma, \operatorname{ad}\overline{r}) = \sum_i (-1)^i \dim_k H^i(\Gamma, \operatorname{ad}\overline{r}),$$

and

$$\chi(\Delta_q, \operatorname{ad}\overline{r}) = \sum_i (-1)^i \dim_k H^i(\Delta_q, \operatorname{ad}\overline{r}),$$

and

$$\chi_{\mathcal{S},T}(\Gamma, \operatorname{ad}\overline{r}) = \sum_i (-1)^i \dim_k H^i_{\mathcal{S},T}(\Gamma, \operatorname{ad}\overline{r}).$$

Then

$$\chi_{\mathcal{S},T}(\Gamma, \operatorname{ad}\overline{r}) = \chi(\Gamma, \operatorname{ad}\overline{r}) - \sum_{q\in S} \chi(\Delta_q, \operatorname{ad}\overline{r}) + \sum_{q\in S-T} (\dim_k L_q - \dim_k H^0(\Delta_q, \operatorname{ad}\overline{r})).$$

The next result is a variant of well known results for GL_n.

Proposition 2.2.9 *Keep the above notation and assumptions, and also assume that $\overline{r}$ is Schur. Then $\operatorname{Def}^{\square_T}_{\mathcal{S}}$ is represented by an object $R^{\square_T}_{\mathcal{S}}$ of $\mathcal{C}_{\mathcal{O}}$. (If $T=\emptyset$ we will denote it $R^{\mathrm{univ}}_{\mathcal{S}}$, while if $T=S$ then we will denote it $R^{\square}_{\mathcal{S}}$.)*

1. *There is a tautological morphism*

$$\widehat{\bigotimes}_{q\in T} R^{\mathrm{loc}}_q/\mathcal{I}_q \longrightarrow R^{\square_T}_{\mathcal{S}}$$

and a canonical isomorphism

$$\operatorname{Hom}_{\mathrm{cts}}(\mathfrak{m}_{R^{\square_T}_{\mathcal{S}}}/(\mathfrak{m}^2_{R^{\square_T}_{\mathcal{S}}}, \lambda, \mathfrak{m}_{R^{\mathrm{loc}}_q})_{q\in T}, k) \cong H^1_{\mathcal{S},T}(\Gamma, \operatorname{ad}\overline{r}).$$

If $H^1(\Gamma, \operatorname{ad}\overline{r})$ is finite dimensional then $R^{\square_T}_{\mathcal{S}}$ is a complete local noetherian $\mathcal{O}$-algebra.

2. *The choice of a universal lifting $r^{\mathrm{univ}}_{\mathcal{S}} : \Gamma \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}})$ determines an extension of the tautological map*

$$R^{\mathrm{univ}}_{\mathcal{S}} \longrightarrow R^{\square_T}_{\mathcal{S}}$$

to an isomorphism

$$R^{\mathrm{univ}}_{\mathcal{S}}[[X_{q,i,j}]]_{q\in T;\, i,j=1,\ldots,n} \xrightarrow{\sim} R^{\square_T}_{\mathcal{S}}.$$

Proof: First we consider representability. By properties 1, 2, 3 and 4 of $\mathcal{D}_q$ we see that the functor sending R to the set of all T-framed lifts of $\overline{r}$ to R of type $\mathcal{S}$ is representable. By property 5 we see that $\mathrm{Def}_{\mathcal{S}}^{\square_T}$ is the quotient of this functor by the smooth group valued functor $R \mapsto \ker(GL_n(R) \to GL_n(k))$. Thus by [Dic] it suffices to check that if $\phi : R \twoheadrightarrow R'$ in $\mathcal{C}_{\mathcal{O}}$, if $(r; \alpha_q)_{q \in T}$ is a T-framed lift of $\overline{r}$ to R, and if $g \in 1 + M_n(\mathfrak{m}_{R'})$ takes $\phi(r; \alpha_q)_{q \in T}$ to itself, then there is a lift $\tilde{g}$ of g in $1 + M_n(\mathfrak{m}_R)$ which takes $(r; \alpha_q)_{q \in T}$ to itself. In the case $T \neq \emptyset$ this is clear, in the case $T = \emptyset$ it follows from lemma 2.1.11.

Recall that

$$\mathrm{Hom}_{\mathrm{cts}}(\mathfrak{m}_{R_{\mathcal{S}}^{\square_T}}/(\mathfrak{m}^2_{R_{\mathcal{S}}^{\square_T}}, \lambda, \mathfrak{m}_{R_q^{\mathrm{loc}}})_{q \in T}, k) \cong \mathrm{Hom}\,(R_{\mathcal{S}}^{\square_T}/(\mathfrak{m}_{R_q^{\mathrm{loc}}})_{q \in T}, k[\epsilon]/(\epsilon^2))$$

is isomorphic to the subspace of $\mathrm{Def}_{\mathcal{S}}^{\square_T}(k[\epsilon]/(\epsilon^2))$ consisting of elements giving trivial liftings of $\overline{r}|_{\Delta_q}$ for $q \in T$. Any T-framed lifting of $\overline{r}$ is of the form

$$((1_n + \phi\epsilon)\overline{r}; 1_n + a_q\epsilon)_{q \in T}$$

with $\phi \in Z^1(\Gamma, \mathrm{ad}\,\overline{r})$. It is of type $\mathcal{S}$ if $\phi|_{\Delta_q} \in L_q^1$ for $q \in S$. For $q \in T$, it gives rise to a trivial lifting of $\overline{r}|_{\Delta_q}$ if and only if

$$(1_n - a_q\epsilon)(1_n + \phi|_{\Delta_q}\epsilon)\overline{r}|_{\Delta_q}(1_n + a_q\epsilon) = \overline{r}|_{\Delta_q}.$$

Thus

$$\mathrm{Hom}_{\mathrm{cts}}(\mathfrak{m}_{R_{\mathcal{S}}^{\square_T}}/(\mathfrak{m}^2_{R_{\mathcal{S}}^{\square_T}}, \lambda, \mathfrak{m}_{R_q^{\mathrm{loc}}})_{q \in T}, k)$$

is in bijection with the set of equivalence classes of tuples

$$(\phi; a_q)_{q \in T}$$

where $\phi \in Z^1(\Gamma, \mathrm{ad}\,\overline{r})$; $a_q \in \mathrm{ad}\,\overline{r}$;

$$\phi|_{\Delta_q} = (\mathrm{ad}\,\overline{r}|_{\Delta_q} - 1_n)a_q$$

for all $q \in T$; and $\phi|_{\Delta_q} \in L_q^1$ for $q \in S - T$. Two tuples $(\phi; a_q)_{q \in T}$ and $(\phi'; a'_q)_{q \in T}$ are equivalent if there exists $b \in \mathrm{ad}\,\overline{r}$ with

$$\phi' = \phi + (1_n - \mathrm{ad}\,\overline{r})b$$

and

$$a'_q = a_q + b$$

for all $q \in T$. The first part of the proposition follows.

Note that by lemma 2.1.11 the centraliser in $1_n + M_n(\mathfrak{m}_{R_{\mathcal{S}}^{\mathrm{univ}}})$ of $r_{\mathcal{S}}^{\mathrm{univ}}$ is $\{1_n\}$. Thus

$$(r_{\mathcal{S}}^{\mathrm{univ}}; 1_n + (X_{q,i,j})_{i,j=1,\ldots,n})_{q \in S}$$

is a universal framed deformation of $\overline{r}$ over $R_{\mathcal{S}}^{\mathrm{univ}}[[X_{q,i,j}]]_{q \in S;\, i,j=1,\ldots,n}$. The second part of the proposition follows. $\square$

Definition 2.2.10 *We will use the following abreviations:*

$$R^{\mathrm{loc}}_{\mathcal{S},T} = \widehat{\bigotimes}_{q\in T} R^{\mathrm{loc}}_q/\mathcal{I}_q$$

and

$$\mathcal{T}_T = \mathcal{O}[[X_{q,i,j}]]_{q\in T;\, i,j=1,\ldots,n}.$$

Thus we have a canonical map

$$R^{\mathrm{loc}}_{\mathcal{S},T} \longrightarrow R^{\square_T}_{\mathcal{S}}$$

and the choice of a universal lifting $r^{\mathrm{univ}}_{\mathcal{S}} : \Gamma \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}})$ *determines a map*

$$\mathcal{T}_T \longrightarrow R^{\square_T}_{\mathcal{S}}$$

such that

$$R^{\mathrm{univ}}_{\mathcal{S}} = R^{\square_T}_{\mathcal{S}}/(X_{q,i,j})_{q\in T;\, i,j=1,\ldots,n}.$$

Lemma 2.2.11 *Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and that I is a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose that $(r;\alpha_q)_{q\in T}$ is a T-framed lifting of $\overline{r}$ to R/I of type $\mathcal{S}$. Suppose moreover that for each $q \in T$ (resp. $q \in S - T$) we are given a lifting $\widehat{r}_q$ of $\alpha_q^{-1} r|_{\Delta_q}\alpha_q$ (resp. $r|_{\Delta_q}$) to R in $\mathcal{D}_q$. For each $\gamma \in \Gamma$ pick a lifting $\widetilde{r(\gamma)}$ of $r(\gamma)$ to $\mathcal{G}_n(R)$. For each $q \in T$ pick a lifting $\widetilde{\alpha}_q$ of a_q to $1_n + M_n(\mathfrak{m}_R)$. Set*

$$\phi(\gamma,\delta) = \widetilde{r(\gamma\delta)}\,\widetilde{r(\delta)}^{-1}\,\widetilde{r(\gamma)}^{-1} - 1_n$$

For $q \in T$ (resp. $q \in S - T$) and $\delta \in \Delta_q$, set

$$\psi_q(\delta) = \widetilde{\alpha}_q^{-1}\widetilde{r(\delta)}\widetilde{\alpha}_q \widehat{r}(\delta)^{-1} - 1_n$$

(resp.

$$\psi_q(\delta) = \widetilde{r(\delta)}\widehat{r}(\delta)^{-1} - 1_n).$$

Then $(\phi,(\psi_q))_{q\in S}$ defines a class $\mathrm{obs}_{\mathcal{S},R,I}(r;\alpha_q)_{q\in T} \in H^2_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) \otimes I$ which is independent of the various choices and vanishes if and only if $(r,\alpha_q)_{q\in T}$ has a T-framed lifting $(\widetilde{r},\widetilde{\alpha}_q)_{q\in T}$ of type $\mathcal{S}$ to R with

$$\widetilde{\alpha}_q^{-1}\widetilde{r}|_{\Delta_q}\widetilde{\alpha}_q = \widehat{r}_q$$

for all $q \in T$.

Proof: We leave the proof to the reader. $\square$

Corollary 2.2.12 *Suppose that $\overline{r}$ is Schur and $H^i(\Gamma, \mathrm{ad}\,\overline{r})$ is finite dimensional for $i \leq 2$. Then $R_{\mathcal{S}}^{\square_T}$ is the quotient of a power series ring in*

$$\dim_k H^1_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r})$$

variables over $R^{\mathrm{loc}}_{\mathcal{S},T}$. If $\mathcal{D}_q$ is liftable for $q \in S - T$ then it will suffice to quotient out by

$$\dim_k H^2_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r})$$

relations and so $R_{\mathcal{S}}^{\square_T}$ has Krull dimension at least

$$\dim_k H^1_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) - \dim_k H^2_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) + 1 + \sum_{q \in T} (\dim R_q^{\mathrm{loc}}/\mathcal{I}_q - 1).$$

Moreover $R_{\mathcal{S}}^{\mathrm{univ}}$ has Krull dimension at least

$$\dim_k H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) - \dim_k H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) + 1 + \sum_{q \in S} (\dim R_q^{\mathrm{loc}}/\mathcal{I}_q - n^2 - 1).$$

Corollary 2.2.13 *Suppose that $\overline{r}$ is Schur, that $H^2_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r}) = (0)$ and that each $\mathcal{D}_q$ is liftable for $q \in S - T$. Then $R_{\mathcal{S}}^{\square_T}$ is a power series ring in $\dim_k H^1_{\mathcal{S},T}(\Gamma, \mathrm{ad}\,\overline{r})$ variables over $R^{\mathrm{loc}}_{\mathcal{S},T}$.*

Finally in this section we turn to a slightly different type of result. Suppose that that $\overline{r}$ is Schur and $\alpha : R_{\mathcal{S}}^{\mathrm{univ}} \twoheadrightarrow \mathcal{O}$ corresponds to a deformation $[r]$ of $\overline{r}$ to $\mathcal{O}$. Let $H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\, r \otimes K/\mathcal{O})$ denote the kernel of

$$H^1(\Gamma, \mathrm{ad}\, r \otimes K/\mathcal{O}) \longrightarrow \bigoplus_{q \in S} H^1(\Delta_q, \mathrm{ad}\, r \otimes K/\mathcal{O})/L_q(r_q).$$

The next lemma is immediate.

Lemma 2.2.14 *Keep the notation and assumptions of the previous paragraph. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{O}}(\ker \alpha/(\ker \alpha)^2, K/\mathcal{O}) \cong H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\, r \otimes_{\mathcal{O}} K/\mathcal{O}).$$

2.3. *Galois deformation theory.* — In this section we specialise some of the results of the previous section to the case of Galois groups.

Let l, k, K, $\mathcal{O}$ and λ be as at the start of the previous section. We will let ϵ denote the l-adic cyclotomic character and write $M(n)$ for $M \otimes_{\mathbf{Z}_l} \mathbf{Z}_l(\epsilon^n)$. Also let ζ_m denote a primitive m^{th} root of unity. We will consider a totally real field F^+ and a totally imaginary quadratic extension F/F^+ split at all places above l. Let S denote a finite set of finite places of F^+ which split in F, and let $F(S)/F$ denote the maximal extension unramified outside S

(and infinity). We will suppose that S contains all the primes of F^+ above l. For Γ we will consider the group $G_{F^+,S} = \mathrm{Gal}\,(F(S)/F^+)$ and for Δ the group $G_{F,S} = \mathrm{Gal}\,(F(S)/F)$. Note that $F(S)/F^+$ may ramify at some places outside S which ramify in F/F^+. If $v|\infty$ is a place of F^+ we will write c_v for some element of the corresponding conjugacy class of complex conjugations in $G_{F^+,S}$. For each $v \in S$ choose a place $\widetilde{v}$ of F above v and let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$. (Thus $\widetilde{S}$ and S are in bijection with each other.) If $v \in S$ then for Δ_v we will consider

$$G_{F_{\widetilde{v}}} = \mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}) \longrightarrow G_{F,S}.$$

(Note that $G_{F_{\widetilde{v}}} \cong \mathrm{Gal}\,(\overline{F}_v^+/F_v^+)$, but the $G_{F,S}$ conjugacy class of the map to $G_{F,S}$ depends on the choice of $\widetilde{v}|v$.) We will write $I_{F_{\widetilde{v}}}$ for the inertia subgroup of $G_{F_{\widetilde{v}}}$ and $\mathrm{Frob}_{\widetilde{v}}$ for the geometric Frobenius in $G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}$.

Let

$$\overline{r} : G_{F^+,S} \longrightarrow \mathcal{G}_n(k)$$

be a continuous homomorphism such that $G_{F,S} = \overline{r}^{-1}(GL_n \times GL_1)(k)$. Let $\chi : G_{F^+,S} \to \mathcal{O}^\times$ a continuous lift of $\nu \circ \overline{r}$. For $v \in S$ let $\mathcal{D}_v$ be a local deformation problem for $\overline{r}|_{G_{F_{\widetilde{v}}}}$. To it we have associated a subspace $L_v \subset H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$ and an ideal $\mathcal{I}_v$ of R_v^{loc}. Together this data defines a global deformation problem for $\overline{r}$ which we will denote

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S}).$$

We will write $L_v^\perp$ for the annihilator in $H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}(1))$ of the subspace L_v of $H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$ under the local duality induced by the pairing

$$\begin{array}{rcl} \mathrm{ad}\,\overline{r} \times (\mathrm{ad}\,\overline{r})(1) & \longrightarrow & k(1) \\ (x,y) & \longmapsto & \mathrm{tr}\,(xy). \end{array}$$

If $T \subset S$ will also write $H^1_{\mathcal{L}^\perp,T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))$ for the kernel of the map

$$H^1(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) \longrightarrow \bigoplus_{v\in S-T} H^1(G_{F_v}\mathrm{ad}\,\overline{r}(1))/L_v^\perp.$$

The next lemma is immediate.

Lemma 2.3.1 *Suppose that*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S})$$

is a deformation problem as above. Suppose also that $S' \supset S$ is a finite set of primes of F^+ which split in F and that $\widetilde{S}' \supset \widetilde{S}$ consists of one prime of F above each element of S'. Define a deformation problem

$$\mathcal{S}' = (F/F^+, S', \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}'_v\}_{v\in S})$$

where, for $v \in S$ we have $\mathcal{D}'_v = \mathcal{D}_v$, and for $v \in S' - S$ the set $\mathcal{D}_v$ consists of all unramified (i.e. minimal) lifts. If $T \subset S$ then $\mathrm{Def}_{\mathcal{S}}^{\square_T}$ is naturally isomorphic to $\mathrm{Def}_{\mathcal{S}'}^{\square_T}$. If $\overline{r}$ is Schur then $R_{\mathcal{S}}^{\square_T} = R_{\mathcal{S}'}^{\square_T}$.

Lemma 2.3.2 *Suppose that*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S})$$

is a deformation problem as above. Suppose that $R \subset S$ contains only primes v for which

- *$v \not| l$,*
- *$\overline{r}$ is unramified at v,*
- *$\mathcal{D}_v$ consists of all unramified (i.e. minimal) lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$.*

Define a new deformation problem

$$\mathcal{S}' = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}'_v\}_{v\in S})$$

where for $v \in S - R$ we set $\mathcal{D}'_{\widetilde{v}} = \mathcal{D}_{\widetilde{v}}$, and for $v \in R$ we let $\mathcal{D}'_{\widetilde{v}}$ consists of all liftings of $\overline{r}|_{G_{F_{\widetilde{v}}}}$.

Suppose that $\phi : R_{\mathcal{S}}^{\mathrm{univ}} \to \mathcal{O}$ and let ϕ_R denote the composite of ϕ with the natural map $R_{\mathcal{S}'}^{\mathrm{univ}} \twoheadrightarrow R_{\mathcal{S}}^{\mathrm{univ}}$. Also let r_ϕ denote $\phi(r_{\mathcal{S}}^{\mathrm{univ}})$. Then

$$\lg_{\mathcal{O}} \ker \phi_R/(\ker \phi_R)^2 \leq \lg_{\mathcal{O}} \ker \phi/(\ker \phi)^2 + \sum_{v\in R} \lg_{\mathcal{O}} H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\, r_\phi)(\epsilon^{-1})).$$

Proof: As described at the end of section 2.2 a class

$$[\psi] \in H^1_{\mathcal{S}'}(G_{F^+,S}, \mathrm{ad}\, r_\phi \otimes \lambda^{-N}/\mathcal{O})$$

corresponds to a deformation $(1+\psi\epsilon)r_\phi$ of $r_\phi \bmod \lambda^N$. This deformation corresponds to an element of $H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\, r_\phi \otimes \lambda^{-N}/\mathcal{O})$ if and only if $(1+\psi\epsilon)r_\phi$ is unramified at all $v \in R$ if and only if $\psi(I_{F_{\widetilde{v}}}) = 0$ for all $v \in R$. Note that, for $v \in R$, we have

$$\begin{aligned} H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, r_\phi \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}) &= \mathrm{Hom}\,(I_{F_{\widetilde{v}}}, \mathrm{ad}\, r_\phi \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}) \\ &= (\mathrm{ad}\, r_\phi) \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}(\epsilon^{-1}). \end{aligned}$$

Thus we have an exact sequence

$$\begin{aligned} &(0) \longrightarrow H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\, r_\phi \otimes \lambda^{-N}/\mathcal{O}) \longrightarrow H^1_{\mathcal{S}'}(G_{F^+,S}, \mathrm{ad}\, r_\phi \otimes \lambda^{-N}/\mathcal{O}) \longrightarrow \\ &\longrightarrow \textstyle\bigoplus_{v\in R} H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\, r_\phi) \otimes_{\mathcal{O}} (\lambda^{-N}/\mathcal{O})(\epsilon^{-1})). \end{aligned}$$

Taking a direct limit and applying lemma 2.2.14 we then get an exact sequence

$$\begin{aligned} &(0) \longrightarrow \mathrm{Hom}\,(\ker \phi/(\ker \phi)^2, K/\mathcal{O}) \longrightarrow \mathrm{Hom}\,(\ker \phi_R/(\ker \phi_R)^2, K/\mathcal{O}) \longrightarrow \\ &\longrightarrow \textstyle\bigoplus_{v\in R} H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\, r_\phi) \otimes_{\mathcal{O}} K/\mathcal{O}(\epsilon^{-1})) \end{aligned}$$

and the lemma follows. □

We will require a lemma from algebraic number theory, which may be known, but for which we do not know a reference.

Lemma 2.3.3 *Let l be a prime, k an algebraic extension of $\mathbf{F}_l$ and $\mathcal{O}$ the ring of integers of a finite totally ramified extension of the field of fractions of $W(k)$. Let λ denote the maximal ideal of $\mathcal{O}$. Let E/D be a Galois extension of number fields with $l\not|[E:D]$. Let S be a finite set of finite places of D containing all places dividing l, and let $E(S)/E$ be the maximal extension unramified outside S. Thus $E(S)/D$ is Galois. Let M be a finite length $\mathcal{O}$-module with a continuous action of* $\mathrm{Gal}\,(E(S)/D)$. *Then*

$$\begin{aligned}&\lg_{\mathcal{O}} H^1(\mathrm{Gal}\,(E(S)/D), M) - \lg_{\mathcal{O}} H^0(\mathrm{Gal}\,(E(S)/D), M)\\ &- \lg_{\mathcal{O}} H^2(\mathrm{Gal}\,(E(S)/D), M) + \textstyle\sum_{v|\infty} \lg_{\mathcal{O}} H^0(\mathrm{Gal}\,(\overline{D}_v/D_v), M)\\ &= [D:\mathbf{Q}] \lg_{\mathcal{O}} M.\end{aligned}$$

Proof: Note that places outside S may ramify in E/D and hence in $E(S)/D$. Nonetheless, as $l\not|[E:D]$, the lemma may be proved in exactly the same way as the usual global Euler characteristic formula. We sketch the argument.

Firstly one shows that if there is a short exact sequence

$$(0) \to M_1 \to M_2 \to M_3 \to (0)$$

and the theorem is true for two of the terms, then it is also true for the third. To do this one considers the long exact sequences for the cohomology groups $H^i(\mathrm{Gal}\,(E(S)/D), M_j)$ and $H^i(\mathrm{Gal}\,(\overline{D}_v/D_v), M_j)$. The key point is that

$$\mathrm{coker}\,(H^2(\mathrm{Gal}\,(E(S)/D), M_2) \longrightarrow H^2(\mathrm{Gal}\,(E(S)/D), M_3))$$

is isomorphic to

$$\mathrm{coker}\,(\bigoplus_{v|\infty} H^0(\mathrm{Gal}\,(\overline{D}_v/D_v), M_2) \longrightarrow \bigoplus_{v|\infty} H^0(\mathrm{Gal}\,(\overline{D}_v/D_v), M_3)).$$

This follows from the equalities

$$\begin{aligned}H^3(\mathrm{Gal}\,(E(S)/D), M_i) &= H^3(\mathrm{Gal}\,(E(S)/E), M_i)^{\mathrm{Gal}\,(E/D)}\\ &\cong (\textstyle\bigoplus_{w|\infty} H^3(\mathrm{Gal}\,(\overline{E}_w/E_w), M_i))^{\mathrm{Gal}\,(E/D)}\\ &= \textstyle\bigoplus_{v|\infty} H^3(\mathrm{Gal}\,(\overline{D}_v/D_v), M_i)\\ &\cong \textstyle\bigoplus_{v|\infty} H^1(\mathrm{Gal}\,(\overline{D}_v/D_v), M_i).\end{aligned}$$

(See for instance (8.6.13)(ii) of [NSW] for the second isomorphism.) Thus we are reduced to the case that M is a k-module.

Next choose a subfield L of $E(S)$ which contains $E(\zeta_l)$, which is totally imaginary and which is finite and Galois over D. Suppose that M is a $\mathrm{Gal}\,(L/D)$-module. Let $L \supset K \supset D$ and let $R_k(\mathrm{Gal}\,(L/K))$ denote the representation ring for $\mathrm{Gal}\,(L/K)$ acting on finite dimensional k-vector spaces. Define a homomorphism

$$\chi_K : R_k(\mathrm{Gal}\,(L/K)) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathbf{Q}$$

by

$$\begin{aligned} &\chi_K[M] \\ &= \dim_k H^1(\mathrm{Gal}\,(E(S)/K), M) - \dim_k H^0(\mathrm{Gal}\,(E(S)/K), M) \\ &\quad - \dim_k H^2(\mathrm{Gal}\,(E(S)/K), M) + \textstyle\sum_{v|\infty} \dim_k H^0(\mathrm{Gal}\,(\overline{K}_v/K_v), M). \end{aligned}$$

This is well defined by the observation of the previous paragraph. We need to show that

$$\chi_D = [D : \mathbf{Q}] \dim_k .$$

It is easy to check that

$$\chi_D \circ \mathrm{Ind}\,^{\mathrm{Gal}\,(L/D)}_{\mathrm{Gal}\,(L/K)} = \chi_K.$$

As $R_k(\mathrm{Gal}\,(L/D)) \otimes \mathbf{Q}$ is spanned by $\mathrm{Ind}\,^{\mathrm{Gal}\,(L/D)}_{\mathrm{Gal}\,(L/K)} R_k(\mathrm{Gal}\,(L/K))$ as K runs over intermediate fields with L/K cyclic of degree prime to l, it suffices to prove that $\chi_K = [K : \mathbf{Q}] \dim_k$ when K is an intermediate field with L/K cyclic of degree prime to l.

Now assume that $L \supset K \supset D$ with L/K cyclic of degree prime to l. Define

$$\widetilde{\chi}_K : R_k(\mathrm{Gal}\,(L/K)) \longrightarrow R_k(\mathrm{Gal}\,(L/K))$$

by

$$\begin{aligned} \widetilde{\chi}_K[M] = &\textstyle\sum_{v|\infty} [M \otimes \mathrm{Ind}\,^{\mathrm{Gal}\,(L/K)}_{\mathrm{Gal}\,(L_w/K_v)} k] + [H^1(\mathrm{Gal}\,(E(S)/L), M)] \\ &- [H^0(\mathrm{Gal}\,(E(S)/L), M)] - [H^2(\mathrm{Gal}\,(E(S)/L), M)], \end{aligned}$$

where w denotes a place of L above v. This is well defined because L totally imaginary implies $H^3(\mathrm{Gal}\,(E(S)/L), M) = (0)$ (see for instance (8.6.13)(ii) of [NSW]). Note that $\widetilde{\chi}_K([M]) = [M(-1)] \otimes \widetilde{\chi}_K([k(1)])$. Moreover as $l \not| [L : K]$ we see that

$$\chi_K = H^0(\mathrm{Gal}\,(L/K), \ \ ) \circ \widetilde{\chi}_K,$$

so that

$$\chi_K([M]) = H^0(\mathrm{Gal}\,(L/K), [M(-1)] \otimes \widetilde{\chi}_K([k(1)])).$$

Thus it suffices to prove that

$$\widetilde{\chi}_K([k(1)]) = [K : \mathbf{Q}][\mathrm{Ind}\,^{\mathrm{Gal}\,(L/K)}_{\{1\}} k].$$

As $E(S)$ is the maximal extension of L unramified outside S one has the standard formulae

$$[H^0(\mathrm{Gal}\,(E(S)/L), k(1))] = [k(1)]$$

and

$$[H^1(\mathrm{Gal}\,(E(S)/L), k(1))] = [\mathcal{O}_L[1/S]^\times \otimes k(1)] + [\mathrm{Cl}_S(L)[l] \otimes_{\mathbf{F}_l} k]$$

and

$$[H^2(\mathrm{Gal}\,(E(S)/L), k(1))] = [\mathrm{Cl}_S(L) \otimes k] - [k] + \sum_{v\in S}[\bigoplus_{w|v} \mathrm{Br}\,(L_w)[l] \otimes_{\mathbf{F}_l} k],$$

where $\mathrm{Cl}_S(L)$ denotes the S-class group of L (i.e. the quotient of the class group by classes of ideals supported over S) and $\mathrm{Br}\,(L_w)$ denotes the Brauer group of L_w. Using these formulae the proof is easily completed, just as in the case of the usual global Euler characteristic formula. $\Box$

Lemma 2.3.4 *Keep the notation and assumptions of the start of this section.*

1. $H^i_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = (0)$ *for* $i > 3$.
2. $H^0_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r})$ *if* $T = \emptyset$ *and* $= (0)$ *otherwise.*
3. $\dim_k H^3_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))$.
4. $\dim_k H^2_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \dim_k H^1_{\mathcal{L}^\perp,T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))$.
5. $\chi_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \sum_{v|\infty} n(n+\chi(c_v))/2 + \sum_{v\in S-T}(\dim_k H^0(G_{F_v}, \mathrm{ad}\,\overline{r}) - \dim_k L_v)$.
6. $$\begin{aligned} &\dim_k H^1_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \\ &= \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}) + \dim_k H^1_{\mathcal{L}^\perp,T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) \\ &\quad - \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) - \textstyle\sum_{v|\infty} n(n+\chi(c_v))/2 \\ &\quad + \textstyle\sum_{v\in S-T}(\dim_k L_v - \dim_k H^0(G_{F_v}, \mathrm{ad}\,\overline{r})) \end{aligned}$$

where we drop the term $\dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r})$ *if* $T \neq \emptyset$.

Proof: For the first part we use the long exact sequences before lemma 2.2.8, and also the vanishing of $H^i(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = H^i(G_{F,S}, \mathrm{ad}\,\overline{r})^{\mathrm{Gal}\,(F/F^+)}$ and $H^i(G_{F_v}, \mathrm{ad}\,\overline{r})$ for $v \in S$ and $i > 2$. For the second part we use the long exact sequences before lemma 2.2.8.

For the third and fourth parts one compares the exact sequences

$$\begin{array}{ccccc} & & H^1(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \to & \left(\bigoplus_{v\in S-T} H^1(G_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{r})/L_{\tilde{v}}\right) \\ & & & & \oplus \left(\bigoplus_{v\in T} H^1(G_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{r})\right) \\ & & & & \downarrow \\ \bigoplus_{v\in S} H^2(G_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{r}) & \leftarrow & H^2(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \leftarrow & H^2_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \\ \downarrow & & & & \\ H^3_{\mathcal{S},T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \to & (0) & & \end{array}$$

and

$$\begin{array}{ccc} & & H^1(G_{F^+,S},\mathrm{ad}\,\overline{r}) \to \left(\bigoplus_{v\in S-T} H^1(G_{F_{\widetilde{v}}},\mathrm{ad}\,\overline{r})/L_{\widetilde{v}}\right) \\ & & \oplus \left(\bigoplus_{v\in T} H^1(G_{F_{\widetilde{v}}},\mathrm{ad}\,\overline{r})\right) \\ & & \downarrow \\ \bigoplus_{v\in S} H^2(G_{F_{\widetilde{v}}},\mathrm{ad}\,\overline{r}) & \leftarrow H^2(G_{F^+,S},\mathrm{ad}\,\overline{r}) \leftarrow & H^1_{\mathcal{L}^\perp,T}(G_{F^+,S},\mathrm{ad}\,\overline{r}(1))^\vee \\ \downarrow & & \\ H^0(G_{F^+,S},\mathrm{ad}\,\overline{r}(1))^\vee \to & (0). & \end{array}$$

(The latter exact sequence is a consequence of Poitou-Tate global duality and the identifications $H^i(G_{F^+,S},\mathrm{ad}\,\overline{r}) = H^i(G_{F,S},\mathrm{ad}\,\overline{r})^{\mathrm{Gal}\,(F/F^+)}$ for $i=1,2$ and $H^i(G_{F^+,S},(\mathrm{ad}\,\overline{r})(1)) = H^i(G_{F,S},(\mathrm{ad}\,\overline{r})(1))^{\mathrm{Gal}\,(F/F^+)}$ for $i=0,1$.)

The fifth and sixth parts follow from lemma 2.2.8, lemma 2.3.3, the local Euler characteristic formula and lemma 2.1.3. (We remark that by the local Euler characteristic formula we have

$$\sum_{v\in S}\chi(G_{F_v},\mathrm{ad}\,\overline{r}) = n^2[F^+:\mathbf{Q}].)$$

The final part follows from the previous parts. □

Combining this with lemma 2.2.12 we get the following corollary.

Corollary 2.3.5 *Keep the notation and assumptions of the start of this section. Suppose also that $\overline{r}$ is Schur. Then $R_{\mathcal{S}}^{\square_T}$ is the quotient of a power series ring in*

$$\begin{array}{l}\dim_k H^1_{\mathcal{L}^\perp,T}(G_{F^+,S},\mathrm{ad}\,\overline{r}(1)) + \sum_{v\in S-T}(\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}},\mathrm{ad}\,\overline{r})) \\ -\dim_k H^0(G_{F^+,S},\mathrm{ad}\,\overline{r}(1)) - \sum_{v|\infty} n(n+\chi(c_v))/2\end{array}$$

variables over $R^{\mathrm{loc}}_{\mathcal{S},T}$. If one further assumes that $\mathcal{D}_v$ is liftable for $v\in S-T$ then it will suffice to quotient by

$$\dim_k H^1_{\mathcal{L}^\perp,T}(G_{F^+,S},\mathrm{ad}\,\overline{r}(1))$$

relations and so $R_{\mathcal{S}}^{\square}$ has Krull dimension at least

$$\begin{array}{l}1+\sum_{v\in T}(\dim R_v^{\mathrm{loc}}/\mathcal{I}_v - 1) + \sum_{v\in S-T}(\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}},\mathrm{ad}\,\overline{r})) \\ -\dim_k H^0(G_{F^+,S},\mathrm{ad}\,\overline{r}(1)) - \sum_{v|\infty} n(n+\chi(c_v))/2.\end{array}$$

Thus $R_{\mathcal{S}}^{\mathrm{univ}}$ has Krull dimension at least

$$1+\sum_{v\in S}(\dim R_v^{\mathrm{loc}}/\mathcal{I}_v - n^2 - 1) - \dim_k H^0(G_{F^+,S},\mathrm{ad}\,\overline{r}(1)) - \sum_{v|\infty} n(n+\chi(c_v))/2.$$

Corollary 2.3.6 *Keep the notation and assumptions of the start of this section. Suppose also that $\bar{r}$ is Schur, that $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\bar{r}(1)) = (0)$ and that each $\mathcal{D}_v$ is liftable for all $v \in S$. Suppose moreover that for $v \in S$ not dividing l we have*

$$\dim_k L_v = \dim_k H^0(G_{F_v}, \mathrm{ad}\,\bar{r}),$$

while for $v|l$ we have

$$\dim_k L_v = [F_v^+ : \mathbf{Q}_l]n(n-1)/2 + \dim_k H^0(G_{F_v}, \mathrm{ad}\,\bar{r}).$$

Then $\chi(c_v) = -1$ for all $v|\infty$, the cohomology group $H^0(G_{F^+,S}, \mathrm{ad}\,\bar{r}(1)) = (0)$ and $R^{\mathrm{univ}}_{\mathcal{S}} = \mathcal{O}$.

2.4. *Some Galois Local Deformation Problems.* — In this section we specialise the discussion still further by considering some explicit local deformation problems $\mathcal{D}_v$ for $\bar{r}|_{G_{F_{\widetilde{v}}}}$. We will continue to use $\mathcal{I}_v$ to denote the ideal of R_v^{loc} corresponding to $\mathcal{D}_v$ and L_v to denote the subspace of $H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\bar{r})$ corresponding to deformations of $\bar{r}$ to $k[\epsilon]/(\epsilon^2)$ of type $\mathcal{D}_v$.

2.4.1. *Crystalline deformations.* — In this section we suppose that $l = p$ and that $F_{\widetilde{v}}$ is unramified over $\mathbf{Q}_p = \mathbf{Q}_l$. We will also suppose that K contains the image of all $\mathbf{Q}_l$-linear embeddings of fields $F_{\widetilde{v}} \hookrightarrow \overline{K}$.

We first recall a (covariant) version of the theory of Fontaine and Laffaille [FL], which will play the key role in this section. Let $\mathrm{Fr} : \mathcal{O}_{F_{\widetilde{v}}} \to \mathcal{O}_{F_{\widetilde{v}}}$ denote the arithmetic Frobenius. Let $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ denote the category of finite $\mathcal{O}_{F_{\widetilde{v}}} \otimes_{\mathbf{Z}_l} \mathcal{O}$-modules M together with

– a decreasing filtration $\mathrm{Fil}^i M$ by $\mathcal{O}_{F_{\widetilde{v}}} \otimes_{\mathbf{Z}_l} \mathcal{O}$-submodules which are $\mathcal{O}_{F_{\widetilde{v}}}$ direct summands with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{l-1} M = (0)$;

– and $\mathrm{Fr} \otimes 1$-linear maps $\Phi^i : \mathrm{Fil}^i M \to M$ with $\Phi^i|_{\mathrm{Fil}^{i+1} M} = l\Phi^{i+1}$ and $\sum_i \Phi^i \mathrm{Fil}^i M = M$.

Let $\mathcal{MF}_{k,\widetilde{v}}$ denote the full subcategory of objects killed by λ. There is an exact, fully faithful, covariant functor of $\mathcal{O}$-linear categories $\mathbf{G}_{\widetilde{v}}$ from $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ to the category of finite $\mathcal{O}$-modules with a continuous action of $G_{F_{\widetilde{v}}}$. Its essential image is closed under taking sub-objects and quotients. The length of M as an $\mathcal{O}$-module is $[k(\widetilde{v}) : \mathbf{F}_l]$ times the length of $\mathbf{G}_{\widetilde{v}}(M)$ as an $\mathcal{O}$-module. (Here $k(\widetilde{v})$ denotes the residue field of $\widetilde{v}$.) For any objects M and N of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ (resp. $\mathcal{MF}_{k,\widetilde{v}}$), the map

$$\mathrm{Ext}^1_{\mathcal{MF}_{\mathcal{O},\widetilde{v}}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}[G_{F_{\widetilde{v}}}]}(\mathbf{G}_{\widetilde{v}}(M), \mathbf{G}_{\widetilde{v}}(N))$$

(resp.

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{MF}_{k,\widetilde{v}}}(M,N) &\longrightarrow \mathrm{Ext}^1_{k[G_{F_{\widetilde{v}}}]}(\mathbf{G}_{\widetilde{v}}(M), \mathbf{G}_{\widetilde{v}}(N)) \\ &\cong\ H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathbf{G}_{\widetilde{v}}(M), \mathbf{G}_{\widetilde{v}}(N))))\end{aligned}$$

is an injection. Moreover

$$\mathrm{Hom}_{\mathcal{MF}_{\mathcal{O},\widetilde{v}}}(M,N) \xrightarrow{\sim} H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_{\mathcal{O}}(\mathbf{G}_{\widetilde{v}}(M), \mathbf{G}_{\widetilde{v}}(N))).$$

We explain how to define $\mathbf{G}_{\widetilde{v}}$ in terms of the functor U_S of [FL]. First we define a contravariant functor

$$\mathrm{Hom}\,(\quad, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})$$

from $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ to itself. Then we set

$$\mathbf{G}_{\widetilde{v}}(M) = U_S(\mathrm{Hom}\,(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})))(2-l).$$

If M is an object of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ we define $\mathrm{Hom}\,(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})) \in \mathcal{MF}_{\mathcal{O},\widetilde{v}}$ as follows.

- The underlying $\mathcal{O}$-module is $\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$.
- $\mathrm{Fil}^a\mathrm{Hom}\,(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})) = \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M/\mathrm{Fil}^{l-1-a}M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$.
- If $f \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M/\mathrm{Fil}^{l-1-a}M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$ and if $m \in \Phi^b\mathrm{Fil}^bM$ set

$$\Phi^a(f)(m) = l^{l-2-a-b}\mathrm{Fr}\,f(\Phi^b)^{-1}(m)).$$

To check that $\Phi^a f$ is well defined one uses the exact sequence

$$\begin{array}{ccccccc} (0) \to \bigoplus_{i=1}^{l-2}\mathrm{Fil}^iM & \to & \bigoplus_{i=0}^{l-2}\mathrm{Fil}^iM & \to & M & \to (0) \\ (m_i) & \mapsto & (lm_i - m_{i+1})_i & & & \\ & & (m_i) & \mapsto & \sum \Phi^i m_i. & \end{array}$$

To check that

$$\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}) = \sum_a \Phi^a\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M/\mathrm{Fil}^{l-1-a}M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$$

it suffices to check that

$$\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}) = \sum_a \Phi^a\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M[l]/\mathrm{Fil}^{l-1-a}M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}).$$

But $M[l] = \bigoplus_i \Phi^i\mathrm{gr}^iM[l]$ and $\Phi^a\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M[l]/\mathrm{Fil}^{l-1-a}M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}) = \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(\Phi^{l-2-a}\mathrm{gr}^{l-2-a}M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$.

In this section we will assume that $\overline{r}$ is in the image of $\mathbf{G}_{\widetilde{v}}$ and that for each i and each $\widetilde{\tau}: F_{\widetilde{v}} \hookrightarrow K$ we have

$$\dim_k(\mathrm{gr}^i\mathbf{G}_{\widetilde{v}}^{-1}(\overline{r}|_{G_{F_{\widetilde{v}}}})) \otimes_{\mathcal{O}_{F_{\widetilde{v}}},\widetilde{\tau}} \mathcal{O} \le 1.$$

We will let $\mathcal{D}_{\widetilde{v}}$ consist of all lifts $r: G_{F_{\widetilde{v}}} \to GL_n(R)$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ such that, for each Artinian quotient R' of R, $r \otimes_R R'$ is in the essential image of $\mathbf{G}_{\widetilde{v}}$. It is easy to verify that this is a local deformation problem and that $L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}})$ will be the image of

$$\mathrm{Ext}^{1}_{\mathcal{MF}_{k,\widetilde{v}}}(\mathbf{G}_{\widetilde{v}}^{-1}(\overline{r}), \mathbf{G}_{\widetilde{v}}^{-1}(\overline{r})) \hookrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}).$$

(This was first observed by Ramakrishna [Ra1].)

Lemma 2.4.1 *$\mathcal{D}_{\widetilde{v}}$ is liftable.*

Proof: Suppose that R is an Artinian object of $\mathcal{C}_{\mathcal{O}}$ and I is an ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that r is a deformation in $\mathcal{D}_{\widetilde{v}}$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ to R/I. Write $M = \mathbf{G}_{\widetilde{v}}^{-1}(r)$ and for $\widetilde{\tau} : F_{\widetilde{v}} \hookrightarrow K$ write $M_{\widetilde{\tau}} = M \otimes_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}, \widetilde{\tau}\otimes 1} \mathcal{O}$. Then $\mathrm{Fil}^i M = \bigoplus_{\widetilde{\tau}} \mathrm{Fil}^i M_{\widetilde{\tau}}$ for all i. Let $m_{\widetilde{\tau},0} < ... < m_{\widetilde{\tau},n-1}$ denote the indices i for which $\mathrm{Fil}^i M_{\widetilde{\tau}} \neq \mathrm{Fil}^{i+1} M_{\widetilde{\tau}}$. Also set $m_{\widetilde{\tau},n} = \infty$ and $m_{\widetilde{\tau},-1} = -\infty$.

As $M/\mathfrak{m}_R M = \mathbf{G}_{\widetilde{v}}^{-1}(\overline{r})$ we see that we can find a surjection $(R/I)^n \to\to M_{\widetilde{\tau}}$ such that $(R/I)^i \twoheadrightarrow \mathrm{Fil}^{m_{\widetilde{\tau},n-i}} M_{\widetilde{\tau}}$ for all i (where $(R/I)^i \subset (R/I)^n$ consists of vectors whose last $n-i$ entries are zero). Counting orders we see that $(R/I)^n \xrightarrow{\sim} M_{\widetilde{\tau}}$, and hence $(R/I)^i \xrightarrow{\sim} \mathrm{Fil}^{m_{\widetilde{\tau},n-i}} M_{\widetilde{\tau}}$ for all i. Define an object $N = \bigoplus_{\widetilde{\tau}} N_{\widetilde{\tau}}$ of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ with an action of R as follows. We take $N_{\widetilde{\tau}} = R^n$ with an $\mathcal{O}_{F,\widetilde{v}}$-action via $\widetilde{\tau}$. We set $\mathrm{Fil}^j N_{\widetilde{\tau}} = R^i$ where $m_{\widetilde{\tau},n-i} \geq j > m_{\widetilde{\tau},n-1-i}$. Then $N/I \cong M$ as filtered $\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} R$-modules. Finally we define $\Phi^{m_{\widetilde{\tau},i}} : \mathrm{Fil}^{m_{\widetilde{\tau},i}} N_{\widetilde{\tau}} \to N_{\widetilde{\tau}\circ\mathrm{Frob}_l}$ by reverse recursion on i. For $i = n-1$ we take any lift of $\Phi^{m_{\widetilde{\tau},n-1}} : \mathrm{Fil}^{m_{\widetilde{\tau},n-1}} M_{\widetilde{\tau}} \to M_{\widetilde{\tau}\circ\mathrm{Frob}_l}$. In general we choose any lift of $\Phi^{m_{\widetilde{\tau},i}} : \mathrm{Fil}^{m_{\widetilde{\tau},i}} M_{\widetilde{\tau}} \to M_{\widetilde{\tau}\circ\mathrm{Frob}_l}$ which restricts to $l^{m_{\widetilde{\tau},i+1} - m_{\widetilde{\tau},i}} \Phi^{m_{\widetilde{\tau},i+1}}$ on $\mathrm{Fil}^{m_{\widetilde{\tau},i+1}} N_{\widetilde{\tau}}$. This is possible as $\mathrm{Fil}^{m_{\widetilde{\tau},i+1}} M_{\widetilde{\tau}}$ is a direct summand of $\mathrm{Fil}^{m_{\widetilde{\tau},i}} M_{\widetilde{\tau}}$. Nakayama's lemma tells us that $\sum_i \Phi^{m_{\widetilde{\tau},i}} \mathrm{Fil}^{m_{\widetilde{\tau},i}} N_{\widetilde{\tau}} = N_{\widetilde{\tau}\circ\mathrm{Frob}_l}$, so that N is an object of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$. As our lifting of r we take $\mathbf{G}_{\widetilde{v}}(N)$. $\square$

We will need to calculate $\dim_k L_{\widetilde{v}}$. To this end we have the following lemma.

Lemma 2.4.2 *Suppose that M and N are objects of $\mathcal{MF}_{k,\widetilde{v}}$. Then there is an exact sequence*

$$\begin{aligned}(0) \to \mathrm{Hom}_{\mathcal{MF}_{k,\widetilde{v}}}(M,N) \to \mathrm{Fil}^0\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}\otimes_{\mathbf{Z}_l}\mathcal{O}}(M,N) \to \\ \to \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}\otimes_{\mathbf{Z}_l}\mathcal{O},\mathrm{Fr}\otimes 1}(\mathrm{gr}\, M, N) \to \mathrm{Ext}^1_{\mathcal{MF}_{k,\widetilde{v}}}(M,N) \to (0),\end{aligned}$$

where $\mathrm{Fil}^i\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}\otimes_{\mathbf{Z}_l}\mathcal{O}}(M,N)$ denotes the subset of $\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}\otimes_{\mathbf{Z}_l}\mathcal{O}}(M,N)$ consisting of elements which take $\mathrm{Fil}^j M$ to $\mathrm{Fil}^{i+j} N$ for all j and where $\mathrm{gr}\, M = \bigoplus_i \mathrm{gr}^i M$. The central map sends β to $(\beta\Phi^i_M - \Phi^i_N\beta)$.

Proof: Any extension

$$(0) \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow (0)$$

in $\mathcal{MF}_{k,\widetilde{v}}$ can be written $E = N \oplus M$ such that $\mathrm{Fil}^i E = \mathrm{Fil}^i N \oplus \mathrm{Fil}^i M$ (and such that $N \to E$ is the natural inclusion and $E \to M$ is the natural projection). Then

$$\Phi^i_E = \begin{pmatrix} \Phi^i_N & \alpha_i \\ 0 & \Phi^i_M \end{pmatrix}$$

with $\alpha_i \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}^i M, N)$. Conversely, any

$$\alpha = (\alpha_i) \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}\, M, N)$$

gives rise to such an extension. Two such extensions corresponding to α and α' are isomorphic if there is a $\beta \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}}(M, N)$ which preserves the filtrations and such that for all i

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi^i_N & \alpha_i \\ 0 & \Phi^i_M \end{pmatrix} = \begin{pmatrix} \Phi^i_N & \alpha'_i \\ 0 & \Phi^i_M \end{pmatrix} \begin{pmatrix} 1 & \beta|_{\mathrm{gr}^i M} \\ 0 & 1 \end{pmatrix}.$$

The lemma now follows easily. $\Box$

Corollary 2.4.3 *Keep the above notation. We have*

$$\dim_k L_{\widetilde{v}} - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}) = [F_{\widetilde{v}} : \mathbf{Q}_l] n(n-1)/2.$$

Hence $R^{\mathrm{loc}}_v / \mathcal{I}_v$ is a power series ring over $\mathcal{O}$ in

$$n^2 + [F_{\widetilde{v}} : \mathbf{Q}_l] n(n-1)/2$$

variables.

Proof: If M is an object of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ and if $\widetilde{\tau} : F_{\widetilde{v}} \hookrightarrow K$ set

$$M_{\widetilde{\tau}} = M \otimes_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}, \tau \otimes 1} \mathcal{O}.$$

Thus $\mathrm{Fil}^i M = \bigoplus_{\widetilde{\tau}} \mathrm{Fil}^i M_{\widetilde{\tau}}$ and $\Phi^i : \mathrm{Fil}^i M_{\widetilde{\tau}} \to M_{\widetilde{\tau} \circ \mathrm{Fr}^{-1}}$. We have

$$\mathrm{Fil}^0 \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}}(M, N) \cong \bigoplus_{\widetilde{\tau}} \mathrm{Fil}^0 \mathrm{Hom}_{\mathcal{O}}(M_{\widetilde{\tau}}, N_{\widetilde{\tau}})$$

and

$$\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbf{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}\, M, N) \cong \bigoplus_{\widetilde{\tau}} \mathrm{Hom}_{\mathcal{O}}(\mathrm{gr}\, M_{\widetilde{\tau}}, N_{\widetilde{\tau} \circ \mathrm{Fr}^{-1}}).$$

Note that $\dim_k \mathrm{Fil}^0 \mathrm{Hom}_k(\mathbf{G}_{\widetilde{v}}^{-1}(\overline{r})_{\widetilde{\tau}}, \mathbf{G}_{\widetilde{v}}^{-1}(\overline{r})_{\widetilde{\tau}}) = n(n+1)/2$ and that $\dim_k \mathrm{Hom}_k(\mathrm{gr}\, \mathbf{G}_{\widetilde{v}}^{-1}(\overline{r})_{\widetilde{\tau}}, \mathbf{G}_{\widetilde{v}}^{-1}(\overline{r})_{\widetilde{\tau} \circ \mathrm{Fr}^{-1}}) = n^2$. The first part of the corollary follows. The second part follows from the first part, lemma 2.4.1 and the discussion immediately following definition 2.2.2. $\Box$

Corollary 2.4.4 *If $n = 1$ then*

$$L_{\widetilde{v}} = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}).$$

Proof: One checks that $L_{\widetilde{v}} \supset H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$ and then uses the equality of dimensions. $\square$

The next lemma is clear.

Lemma 2.4.5 *If $\overline{r}|_{G_{F_{\widetilde{v}}}} = \oplus_i \overline{s}_i$ then*

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = \oplus_{i,j} H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{s}_i, \overline{s}_j))$$

and $L_{\widetilde{v}} = \oplus_{i,j}(L_{\widetilde{v}})_{i,j}$, where $(L_{\widetilde{v}})_{i,j}$ denotes the image of

$$\mathrm{Ext}^1_{\mathcal{MF}_{k,\widetilde{v}}}(\mathbf{G}_{\widetilde{v}}^{-1}(\overline{s}_i), \mathbf{G}_{\widetilde{v}}^{-1}(\overline{s}_j)) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{s}_i, \overline{s}_j)).$$

2.4.2. *Ordinary deformations.* — This section is not required for our applications to modularity lifting theorems and the Sato-Tate conjecture, and can be skipped by those readers whose only interest is in these applications. Our discussion is rather unsatisfactory as we were unable to find the right degree of generality in which to work. In the first version of this manuscript we worked in greater generality, but the result was so complicated that some of the referees urged us to remove the section all together. Rather than do so we have restricted ourselves to the easiest possible case. We hope that the result is more readable. We also hope that future investigators will either not need to rediscover our messy but more complete results, or that they will be able to find a more transparent approach.

A referee has reminded us of previous work of Tilouine [Ti] and Mauger [Mau] along similar lines.

We again assume that $p = l$. For $i = 0, \ldots, n-1$ choose characters $\chi_{v,i} : G_{F_{\widetilde{v}}} \to \mathcal{O}^\times$ with the following properties.

1. $\overline{r}$ has a decreasing filtration $\{\overline{\mathrm{Fil}}^i\}$ by $k[G_{F_{\widetilde{v}}}]$-submodules such that for $i = 0, ..., n-1$ we have an isomorphism $\overline{\mathrm{gr}}^i\overline{r} \cong k(\chi_{v,i})$.
2. If $\overline{\chi}_{v,i}$ denotes the reduction of $\chi_{v,i}$ modulo λ then for $i < j$ the ratio $\overline{\chi}_{v,i}/\overline{\chi}_{v,j}$ is neither trivial nor the cyclotomic character.

The second of these two conditions can be weakened, but we have not been able to determine exactly how far. Note that the second condition implies that the filtration $\{\overline{\mathrm{Fil}}^i\}$ is unique.

We will take $\mathcal{D}_v$ to be the set of all lifts r of $\overline{r}$ to objects R of $\mathcal{C}_{\mathcal{O}}$ such that R^n has a decreasing filtration $\{\mathrm{Fil}^i\}$ by $R[G_{F_{\widetilde{v}}}]$-submodules such that

1. $\mathrm{Fil}^i \otimes_R k \xrightarrow{\sim} \overline{\mathrm{Fil}}^i$ for all i, and
2. $I_{F_{\widetilde{v}}}$ acts on $\mathrm{gr}^i R^n$ by $\chi_{v,i}$.

It follows from the first of these properties that the Fil^i are free over R and direct summands of R^n. Moreover for $i = 0, ..., n-1$ the graded piece $\mathrm{gr}^i R^n \cong R(\chi_i')$ where χ_i' is an unramified twist of $\chi_{v,i}$ which reduces modulo $\mathfrak{m}_R$ to $\chi_{v,i} \bmod \lambda$.

Lemma 2.4.6 *1. If such a filtration $\{\mathrm{Fil}^i\}$ exists then it is unique.*
2. Suppose that $R \hookrightarrow S$ is an injective morphism in $\mathcal{C}_\mathcal{O}$ and that $r : G_{F_{\widetilde{v}}} \to GL_n(R)$ is a lift of $\overline{r}|_{G_{F_{\widetilde{v}}}}$. If $(S, r) \in \mathcal{D}_v$ and $\{\mathrm{Fil}^i_S\}$ is the corresponding filtration of S^n then

$$(\mathrm{Fil}^i_S \cap R^n) \otimes_R S \xrightarrow{\sim} \mathrm{Fil}^i_S.$$

3. $\mathcal{D}_v$ is a local deformation problem.

Proof: The third part follows from the first two. For the first two parts, arguing inductively it suffices to treat the case of Fil^{n-1}. For $i = 0, ..., n-2$ choose $\sigma_i \in G_{F_{\widetilde{v}}}$ with $\overline{\chi}_{v,i}(\sigma_i) \neq \overline{\chi}_{v,n-1}(\sigma_i)$. Let $P_i(X)$ denote the characteristic polynomial of $r(\sigma_i)$. Modulo $\mathfrak{m}_R$ we have a factorisation

$$P_i(X) \equiv (X - \chi_{v,n-1}(\sigma_i))^{a_i} \overline{Q}_i(X) \bmod \mathfrak{m}_R$$

with $\overline{Q}_i(\chi_{v,n-1}(\sigma_i)) \not\equiv 0 \bmod \mathfrak{m}_R$. By Hensel's lemma we may lift this to a factorisation

$$P_i(X) = R_i(X) Q_i(X)$$

where Q_i lifts $\overline{Q}_i$ and R_i lifts $(X - \overline{\chi}_{v,n-1}(\sigma_i))^{a_i}$. Let

$$e = \prod_{i=0}^{n-2} Q_i(r(\sigma_i)).$$

Then e acts as zero on $\mathrm{gr}^i R^n$ (resp. $\mathrm{gr}^i_S S^n$) for $i = 0, ..., n-2$ (because $Q_i(r(\sigma_i))$ does). On the other hand e is an isomorphism on $\mathrm{Fil}^{n-1} R^n$ (resp. $\mathrm{Fil}^{n-1}_S S^n$), so that $\mathrm{Fil}^{n-1} R^n = eR^n$ (resp. $\mathrm{Fil}^{n-1}_S S^n = eS^n$). The first part follows immediately. In the case of the second part note that $ek^n \neq (0)$. Choose $y \in R^n$ such that the image of ey in k^n is non-zero. Then $\mathrm{Fil}^{n-1}_S S^n = Sey$ so that $\mathrm{Fil}^{n-1}_S \cap R^n = Rey$. The second part of the lemma follows. □

Lemma 2.4.7 *$\mathcal{D}_v$ is liftable.*

Proof: Suppose that R is an object of $\mathcal{C}_\mathcal{O}$ and I is a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that r is a deformation in $\mathcal{D}_v$ of $\overline{r}$ to R/I. Let $\{\mathrm{Fil}^i\}$ be the corresponding filtration of $(R/I)^n$. We will show by reverse induction on i that we can find a lifting $\widetilde{\mathrm{Fil}}^i$ of $\mathrm{Fil}^i r$ to R such that $\widetilde{\mathrm{Fil}}^{i+1} \hookrightarrow \widetilde{\mathrm{Fil}}^i$ compatibly with $\mathrm{Fil}^{i+1} r \hookrightarrow \mathrm{Fil}^i r$ and $\widetilde{\mathrm{Fil}}^i / \widetilde{\mathrm{Fil}}^{i+1} \cong R(\chi_{v,i}|_{I_{F_{\widetilde{v}}}})$ as a $R[I_{F_{\widetilde{v}}}]$-module.

The case $i = n-1$ is trivial. Suppose that $\widetilde{\mathrm{Fil}}^{i+1}$ has been constructed. Also choose a lifting $\widetilde{\mathrm{gr}}^i$ of $\mathrm{gr}^i r$ such that $I_{F_{\widetilde{v}}}$ acts by $\chi_{v,i}$. We will choose $\widetilde{\mathrm{Fil}}^i$ to be an extension of $\widetilde{\mathrm{gr}}^i$ by $\widetilde{\mathrm{Fil}}^{i-1}$ which lifts $\mathrm{Fil}^i r$. Such extensions are parametrised by some fibre of the map

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\widetilde{\mathrm{gr}}^i, \widetilde{\mathrm{Fil}}^{i+1})) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_{R/I}(\mathrm{gr}^i r, \mathrm{Fil}^{i+1} r)).$$

Thus it suffices to show that this map is surjective. This would follow if

$$H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{gr}^i \overline{r}, \mathrm{Fil}^{i+1} \overline{r})) \otimes_k I = (0).$$

However locally duality tells us that $H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{gr}^i \overline{r}, \mathrm{Fil}^{i+1} \overline{r}))$ is dual to $H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{Fil}^{i+1} \overline{r}, \mathrm{gr}^i \overline{r})(1))$, and this latter group vanishes, because, for $j > i$,

$$\overline{\chi}_{v,i} \epsilon / \overline{\chi}_{v,j} \neq 1.$$

□

Lemma 2.4.8 *$R_v^{\mathrm{loc}}/\mathcal{I}_v$ is a power series ring in*

$$n^2 + [F_{\widetilde{v}} : \mathbf{Q}_l] n(n-1)/2$$

variables over $\mathcal{O}$. Moreover

$$\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}) = [F_{\widetilde{v}} : \mathbf{Q}_l] n(n-1)/2.$$

Proof: From the previous lemma and discussion immediately following definition 2.2.2, we see that the two assertions are equivalent. Moreover they are both equivalent to the space of liftings of type $\mathcal{D}_v$ of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ having dimension $n^2 + [F_{\widetilde{v}} : \mathbf{Q}_l] n(n-1)/2$.

Let B_n denote the Borel subgroup of GL_n consisting of upper triangular matrices. Without loss of generality we may suppose that $\overline{r}$ maps $G_{F_{\widetilde{v}}}$ to $B_n(k)$ so that the diagonal entries of $\overline{r}(\sigma)$ reading from the top left are $(\overline{\chi}_{v,n-1}(\sigma), ..., \overline{\chi}_{v,0}(\sigma))$. The space of liftings of type $\mathcal{D}_v$ of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ maps surjectively to the space of filtrations $\{\mathrm{Fil}^i\}$ of $k[\epsilon]/(\epsilon^2)$ lifting $\{\overline{\mathrm{Fil}}^i\}$ with kernel the space of liftings of $\overline{r}$ to $B_n(k[\epsilon]/(\epsilon^2))$ such that for $\sigma \in I_{F_{\widetilde{v}}}$ the element $\overline{r}(\sigma)$ has diagonal entries $(\overline{\chi}_{v,n-1}(\sigma), ..., \overline{\chi}_{v,0}(\sigma))$ reading from the top left. For the rest of this proof, we will call such a lift suitable. Thus it suffices to show the space of suitable lifts to $B_n(k[\epsilon]/(\epsilon^2))$ has dimension $n(n+1)/2 + [F_{\widetilde{v}} : \mathbf{Q}_l]n(n-1)/2$.

We will prove this by induction on n. The case $n = 1$ is clear. (A lift is specified by specifying a lift of any element lying over Frobenius.) For general n write

$$\overline{r} = \begin{pmatrix} \overline{r}' & \overline{\chi}_{v,0}\overline{\psi} \\ 0 & \overline{\chi}_{v,0} \end{pmatrix}.$$

By the argument in the proof of the last lemma we see that the space of suitable lifts of $\overline{r}$ to $B_n(k[\epsilon]/(\epsilon^2))$ maps surjectively to the sum of the space of suitable lifts of $\overline{r}'$ to $B_{n-1}(k[\epsilon]/(\epsilon^2))$ and the space of lifts of $\overline{\chi}_{v,0}$ to $(k[\epsilon]/(\epsilon^2))^\times$ which agree with $\overline{\chi}_{v,0}$ on $I_{F_{\widetilde{v}}}$. Thus by the inductive hypothesis it suffices to show that the set of lifts of $\overline{r}$ to $B_n(k[\epsilon]/(\epsilon^2))$ of the form

$$\begin{pmatrix} \overline{r}' & \overline{\chi}_{v,0}\psi \\ 0 & \overline{\chi}_{v,0} \end{pmatrix}$$

has dimension $(1 + [F_{\widetilde{v}} : \mathbf{Q}_l])(n-1)$.

However the latter space can be identified with a fibre of the surjective linear map:

$$Z^1(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}' \otimes_k k[\epsilon]/(\epsilon^2)) \twoheadrightarrow Z^1(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}').$$

This map has kernel $Z^1(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}')\epsilon$, which has dimension

$$n - 1 + \dim_k H^1(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}') - \dim_k H^0(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}')$$

which (by the local Euler characteristic formula) equals

$$n - 1 + [F_{\widetilde{v}} : \mathbf{Q}_l](n-1) + \dim_k H^2(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}').$$

As we saw in the proof of the last lemma $H^2(G_{F_{\widetilde{v}}}, \overline{\chi}_{v,0}^{-1}\overline{r}') = (0)$ and this lemma follows. □

2.4.3. *Unrestricted deformations.* — Suppose now that $l \neq p$. We can take $\mathcal{D}_v$ to consist of all lifts of $\overline{r}$ in which case $L_v = H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$. In this case, by the local Euler characteristic formula,

$$\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = \dim_k H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r})(1)).$$

Lemma 2.4.9 *If $H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r})(1)) = (0)$ then $H^2(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = (0)$, $\mathcal{D}_v$ is liftable, and R_v^{loc} is a power series ring in n^2 variables over $\mathcal{O}$.*

(In fact these four conditions are probably all equivalent.)

2.4.4. *Minimal deformations.* — Again suppose that $p \neq l$. In this section we will describe certain lifts of $\bar{r}$ which can be considered 'minimally' ramified. We will show that these lifts constitute a liftable, local deformation problem and calculate the dimension of the corresponding $R_v^{\mathrm{loc}}/\mathcal{I}_v$. However first we must discuss a general classification of lifts of $\bar{r}$.

If $q \in \mathbf{Z}_{>0}$ is not divisible by l, we will write T_q for the semidirect product of $\mathbf{Z}_l = \langle \sigma_q \rangle$ by $\widehat{\mathbf{Z}} = \langle \phi_q \rangle$ where ϕ_q acts on $\mathbf{Z}_l$ by multiplication by q.

Let $P_{F_{\widetilde{v}}}$ denote the kernel of any (and hence every) surjection $I_{F_{\widetilde{v}}} \twoheadrightarrow \mathbf{Z}_l$. Then $P_{F_{\widetilde{v}}}$ has pro-order prime to l. Also set $T_{F_{\widetilde{v}}} = G_{F_{\widetilde{v}}}/P_{F_{\widetilde{v}}} \cong T_{\mathbf{N}\widetilde{v}}$.

Lemma 2.4.10 *The exact sequence*

$$(0) \longrightarrow P_{F_{\widetilde{v}}} \longrightarrow G_{F_{\widetilde{v}}} \longrightarrow T_{F_{\widetilde{v}}} \longrightarrow (0)$$

splits, so that $G_{F_{\widetilde{v}}}$ becomes the semidirect product of $P_{F_{\widetilde{v}}}$ by $T_{F_{\widetilde{v}}}$. We will fix one such splitting.

Proof: Let S denote a Sylow pro-l-subgroup of $I_{F_{\widetilde{v}}}$ so that $S \xrightarrow{\sim} \mathbf{Z}_l$. Let ϕ denote a lift to $G_{F_{\widetilde{v}}}$ of $\mathrm{Frob}_{\widetilde{v}}^{-1} \in G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}$. The conjugate $\phi S \phi^{-1}$ is another Sylow pro-l-subgroup of $I_{F_{\widetilde{v}}}$ and hence an $I_{F_{\widetilde{v}}}$-conjugate of S. Thus premultiplying ϕ by an element of $I_{F_{\widetilde{v}}}$ we may suppose that ϕ normalises S. The group topologically generated by S and ϕ maps isomorphically to $T_{F_{\widetilde{v}}}$ and we have our desired splitting. □

Suppose that τ is an irreducible representation of $P_{F_{\widetilde{v}}}$ over k. We will write G_τ for the group of $\sigma \in G_{F_\tau}$ with $\tau^\sigma \sim \tau$. We will also write $T_\tau = G_\tau/P_{F_{\widetilde{v}}} \subset T_{F_{\widetilde{v}}}$. Then $T_\tau \cong T_{(\mathbf{N}\widetilde{v})^{[G_{F_{\widetilde{v}}}:G_\tau I_{F_{\widetilde{v}}}]}}$ and the splitting $T_{F_{\widetilde{v}}} \hookrightarrow G_{F_{\widetilde{v}}}$ restricts to a splitting $T_\tau \hookrightarrow G_\tau$.

The proof of the next lemma uses standard techniques of what is sometimes called Clifford theory (see section 11 of [CR]).

Lemma 2.4.11 *1. $l \not| \dim_k \tau$ and τ has a unique deformation to a representation $\widetilde{\tau}$ of $P_{F_{\widetilde{v}}}$ over $\mathcal{O}$.*

2. τ has a unique (up to equivalence) extension to $G_\tau \cap I_{F_{\widetilde{v}}}$. Moreover $\widetilde{\tau}$ has a unique extension $G_\tau \cap I_{F_{\widetilde{v}}}$ with determinant of order prime to l.

3. $\widetilde{\tau}$ has an extension to G_τ with $\det \widetilde{\tau}(G_\tau \cap I_{F_{\widetilde{v}}})$ having order prime to l. Choose such an extension, which we will also denote $\widetilde{\tau}$, and let τ also denote its reduction modulo λ.

Proof: The first part is true because $P_{F_{\widetilde{v}}}$ has pro-order prime to l.

Any Sylow pro-l-subgroup of $G_\tau \cap I_{F_{\widetilde{v}}}$ maps isomorphically to $G_\tau/(G_\tau \cap I_{F_{\widetilde{v}}})$. Let σ_τ denote a topological generator of a Sylow pro-l-subgroup of $G_\tau \cap I_{F_{\widetilde{v}}}$. The kernel of τ is normal in G_τ. The conjugation action of some power σ^{l^b} of σ_τ on the image $\tau P_{F_{\widetilde{v}}}$ is trivial. Because $\sigma_\tau \in G_\tau$, there is an automorphism A of the vector space underlying τ such that $\tau(\sigma_\tau g \sigma_\tau^{-1}) = A\tau(g)A^{-1}$ for all $g \in P_{F_{\widetilde{v}}}$. Then we see that $z = A^{l^b}$ lies in the centraliser Z_τ of the image of τ. As τ is irreducible we see that Z_τ is the multiplicative group of a finite extension of k and so is a torsion abelian group with order prime to l. Moreover $\mathbf{Z}/l^b\mathbf{Z}$ acts on Z_τ by letting 1 act by conjugation by A. As $H^2(\mathbf{Z}/l^b\mathbf{Z}, Z_\tau) = (0)$ we see that there is $w \in Z_\tau$ with $z^{-1} = w(AwA^{-1})(A^2wA^{-2})...(A^{l^b-1}wA^{1-l^b}) = (wA)^{l^b}A^{-l^b}$. We can extend τ to $G_\tau \cap I_{F_{\widetilde{v}}}$ by sending σ_τ to wA. Now write A for wA. Any other extension sends σ_τ to uA for some $u \in Z_\tau$ with $u(AuA^{-1})...(A^{l^b-1}uA^{1-l^b})$ equalling an element of Z_τ of l-power order, i.e. equalling 1. As $H^1(\mathbf{Z}/l^b\mathbf{Z}, Z_\tau) = (0)$ we see that $u = v^{-1}AvA^{-1}$ for some $v \in Z_\tau$. Hence our second extension of $\tau|_{P_{F_{\widetilde{v}}}}$ is $v^{-1}\tau v$, i.e. our extension is unique up to equivalence. Similarly the lifting $\widetilde{\tau}$ has a unique extension to $G_\tau \cap I_{F_{\widetilde{v}}}$ with determinant of order prime to l. (Argue as before but choose A with $\det A$ having order prime to l, which is possible as for $z \in \mathcal{O}^\times$ we have $\det(zA) = z^{\dim \tau}\det(A)$. Then take Z_τ to be the set of elements of the centraliser of $\widetilde{\tau}(P_{F_{\widetilde{v}}})$ with order prime to l. The same argument shows the existence of one extension with determinant of order prime to l and also its uniqueness.)

Let $\phi_\tau \in G_\tau$ lift a generator of $G_\tau/(G_\tau \cap I_{F_{\widetilde{v}}})$. As $\widetilde{\tau}$ and $\widetilde{\tau}^{\phi_\tau}$ are equivalent as representations of $G_\tau \cap I_{F_{\widetilde{v}}}$, the representation $\widetilde{\tau}$ extends to G_τ. □

If M is a finite $\mathcal{O}$-module with a continuous action of $G_{F_{\widetilde{v}}}$ then we set

$$M_\tau = \mathrm{Hom}_{P_{F_{\widetilde{v}}}}(\widetilde{\tau}, M).$$

It is naturally a continuous T_τ-module.

Lemma 2.4.12 *Suppose that M is a finite $\mathcal{O}$-module with a continuous action of $G_{F_{\widetilde{v}}}$. Then there is a natural isomorphism*

$$M \cong \bigoplus_{[\tau]} \mathrm{Ind}_{G_\tau}^{G_{F_{\widetilde{v}}}}(\widetilde{\tau} \otimes_{\mathcal{O}} M_\tau),$$

where $[\tau]$ runs over $G_{F_{\widetilde{v}}}$-conjugacy classes of irreducible $k[P_{F_{\widetilde{v}}}]$-modules. Moreover

$$\mathrm{Hom}_{G_{F_{\widetilde{v}}}}(M, M') \cong \bigoplus_{[\tau]} \mathrm{Hom}_{T_\tau}(M_\tau, M'_\tau).$$

Proof: We have a decomposition

$$M \cong \bigoplus_{[\tau]} \widetilde{\tau} \otimes_{\mathcal{O}} M_\tau,$$

where $[\tau]$ runs over isomorphism classes of irreducible $k[P_{F_{\widetilde{v}}}]$-modules. The embedding $\widetilde{\tau} \otimes_{\mathcal{O}} M_\tau \hookrightarrow M$ is G_τ-equivariant and the image is the biggest submodule all whose simple $\mathcal{O}[P_{F_{\widetilde{v}}}]$-subquotients are isomorphic to τ. Moreover $\sigma \in G_{F_{\widetilde{v}}}$ takes the image of $\widetilde{\tau} \otimes_{\mathcal{O}} M_\tau$ to $\widetilde{\tau}^\sigma \otimes_{\mathcal{O}} M_{\tau^\sigma}$. The lemma follows. $\square$

Corollary 2.4.13 *Suppose R is an object of $\mathcal{C}_{\mathcal{O}}^f$. The map*

$$r \longmapsto (r_\tau)_{[\tau]}$$

sets up a bijection between deformations r of $\overline{r}$ (as a $G_{F_{\widetilde{v}}}$-representation) to R, and tuples $(r_\tau)_{[\tau]}$ of deformations of $\overline{r}_\tau$ (as T_τ-representations) to R, where $[\tau]$ runs over $G_{F_{\widetilde{v}}}$-conjugacy classes of irreducible $k[P_{F_{\widetilde{v}}}]$-modules.

Definition 2.4.14 *Let $\overline{\rho}$ be an m dimensional representation of T_q over k, and let ρ denote a lifting of $\overline{\rho}$ to an object R of $\mathcal{C}_{\mathcal{O}}^f$. We will call r* minimally ramified *if for all i the natural map*

$$\ker(\rho(\sigma_q) - 1_m)^i \otimes_R k \longrightarrow \ker(\overline{\rho}(\sigma_q) - 1_m)^i$$

is an isomorphism.

We call a lifting r of $\overline{r}$ to a representation of $G_{F_{\widetilde{v}}}$ over an object R of $\mathcal{C}_{\mathcal{O}}^f$ minimally ramified *if, for all irreducible $k[P_{F_{\widetilde{v}}}]$-modules τ, the deformation r_τ of $\overline{r}_\tau$ is minimally ramified (as a representation of T_τ).*

For this definition to make sense we need to make two remarks. Firstly, for any $g \in 1_m + M_m(\mathfrak{m}_R)$ a lifting ρ (resp. r) is minimally ramified if and only if $g\rho g^{-1}$ (resp. grg^{-1}) is. Secondly, in the case of T_q, the definition of minimally ramified does not depend on the choice of generator σ_q of $\mathbf{Z}_l$. (Indeed if σ_q' is another generator of $\mathbf{Z}_l$ then $\rho(\sigma_q') = \rho(\sigma_q)^a$ for some $a \in \mathbf{Z}_{>0}$ not divisible by l. Then $\rho(\sigma_q') - 1_m = (\rho(\sigma_q) - 1_m)(1_m + \rho(\sigma_q) + \ldots + \rho(\sigma_q)^{a-1})$ so that $\ker(\rho(\sigma_q') - 1_m)^i \supset \ker(\rho(\sigma_q) - 1_m)^i$. Similarly $\ker(\rho(\sigma_q) - 1_m)^i \supset \ker(\rho(\sigma_q') - 1_m)^i$, so the two kernels are in fact equal.)

We remark that if $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is unramified then minimally ramified lifts are just unramified lifts.

Lemma 2.4.15 *Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$. Let $A \in M_{m_1 \times m_2}(R)$ and let $\overline{A}$ denote its image in $M_{m_1 \times m_2}(k)$. We can find bases $e_1, \ldots, e_{m_2}$ of R^{m_2} and $f_1, \ldots, f_{m_1}$ of R^{m_1} such that $Ae_i = f_i$ for $i = 1, \ldots, r$ and $Ae_i \in \mathfrak{m}_R f_{r+1} \oplus \ldots \oplus \mathfrak{m}_R f_{m_1}$ for $i = r+1, \ldots, m_2$. Moreover the following are equivalent.*

1. *$Ae_i = 0$ for $i = r+1, \ldots, m_2$.*
2. *$(\ker A) \otimes_R k \xrightarrow{\sim} \ker \overline{A}$.*
3. *$(\ker A) \otimes_R k \twoheadrightarrow \ker \overline{A}$.*

4. $(\operatorname{Im} A) \otimes_R k \xrightarrow{\sim} \operatorname{Im} \overline{A}$.
5. $(\operatorname{Im} A) \otimes_R k \hookrightarrow \operatorname{Im} \overline{A}$.

Proof: Choose a basis $\overline{e}_1, ..., \overline{e}_{m_2}$ of k^{m_2} so that $\overline{e}_{r+1}, ..., \overline{e}_{m_2}$ is a basis of $\ker \overline{A}$. Let $\overline{f}_i = A\overline{e}_i$ for $i = 1, ..., r$ and extend $\overline{f}_1, ..., \overline{f}_r$ to a basis $\overline{f}_1, ..., \overline{f}_{m_1}$ of k^{m_1}. Lift $\overline{e}_1, ..., \overline{e}_{m_2}$ to a basis $e_1, ..., e_r, e'_{r+1}, ..., e'_{m_2}$ of R^{m_2}. Also lift $\overline{f}_1, ..., \overline{f}_{m_1}$ to a basis $f_1 = Ae_1, ..., f_r = Ae_r, f_{r+1}, ..., f_{m_1}$ of R^{m_1}. For $i = r+1, ..., m_2$ write $Ae'_i = \sum_{j=1}^{m_1} a_{ij} f_j$ with each $a_{ij} \in \mathfrak{m}_R$ and set

$$e_i = e'_i - \sum_{j=1}^{r} a_{ij} e_j.$$

Then $e_1, ..., e_{m_2}$ is a basis of R^{m_2} with $Ae_i = f_i$ for $i = 1, ..., r$, while $Ae_i \in \mathfrak{m}_R f_{r+1} \oplus ... \oplus \mathfrak{m}_R f_{m_1}$ for $i = r+1, ...m_2$.

Now consider the second part of the lemma. The first condition implies the second, which implies the third. Suppose the third condition is satisfied. Then $\ker A$ is a submodule of $Re_{r+1} \oplus ... \oplus Re_{m_2}$ which surjects under reduction modulo $\mathfrak{m}_R$ onto $k\overline{e}_{r+1} \oplus ... \oplus \overline{e}_{m_2}$. We deduce that $\ker A = Re_{r+1} \oplus ... \oplus Re_{m_2}$, and the first condition follows.

Similarly the first condition implies the fourth which implies the fifth. Suppose the fifth condition is satisfied. Let $X = A(Re_{r+1} \oplus ... \oplus Re_{m_2})$, so that $\operatorname{Im} A = Rf_1 \oplus ... \oplus Rf_r \oplus X$. We deduce that $X \otimes_R k = (0)$, so that $X = 0$ and the first condition follows. $\Box$

Corollary 2.4.16 *Suppose that $R \to S$ is a morphism in $\mathcal{C}^f_{\mathcal{O}}$ and that $A \in M_{m_1 \times m_2}(R)$ satisfies the conditions of the equivalent conditions of the lemma. Then so does the image of A in $M_{m_1 \times m_2}(S)$.*

Corollary 2.4.17 *Suppose that $R \hookrightarrow S$ is an injective morphism in $\mathcal{C}^f_{\mathcal{O}}$ and that $A \in M_{m_1 \times m_2}(R)$. Suppose that the image of A in $M_{m_1 \times m_2}(S)$ satisfies the equivalent conditions of the lemma, then so does $A \in M_{m_1 \times m_2}(R)$.*

Corollary 2.4.18 *Minimally ramified lifts in the case of T_q (resp. $G_{F_{\tilde{v}}}$) define a local deformation problem $\mathcal{D}^T_v$ (resp. $\mathcal{D}_v$) in the sense of definition 2.2.2.*

We claim that a lifting ρ of an m-dimensional representation $\overline{\rho}$ of T_q over k to R an object of $\mathcal{C}^f_{\mathcal{O}}$ is minimally ramified if and only if there is an increasing filtration $\{\mathrm{Fil}^i\}$ of ρ by T_q-invariant direct summands such that $\rho(\sigma_q)$ acts trivially on each $\mathrm{gr}^i \rho = \mathrm{Fil}^i \rho / \mathrm{Fil}^{i-1} \rho$ and

$$\mathrm{Fil}^i \otimes_R k \xrightarrow{\sim} \ker(\overline{\rho}(\sigma_q) - 1_m)^i$$

under the natural map $\mathrm{Fil}^i \otimes_R k \to \overline{\rho}$. Moreover in this case there is a unique such filtration, namely $\mathrm{Fil}^i = \ker(\rho(\sigma_q) - 1_m)^i$. To see this first note that if ρ

is minimally ramified then it follows from Nakayama's lemma that $\ker(\rho(\sigma_q)-1_m)^i$ is a direct summand of ρ for all i. Conversely if $\{\mathrm{Fil}^i\}$ is a filtration as above then $\ker(\rho(\sigma_q)-1_m)^i \supset \mathrm{Fil}^i\rho$. On the other hand, as the rank of $(\overline{\rho}(\sigma_q)-1_m)^i$ equals m minus the R-rank of $\mathrm{Fil}^i\rho$, we see that we must have equality $\ker(\rho(\sigma_q)-1_m)^i = \mathrm{Fil}^i\rho$ and our claim follows.

Lemma 2.4.19 *Suppose that $\overline{\rho} : T_q \longrightarrow GL_m(k)$ is a continuous representation. The universal minimally ramified lifting ring $R_{\overline{\rho}}^{\min}$ for $\overline{\rho}$ is a power series ring in m^2 variables over $\mathcal{O}$.*

Proof: The filtration $\ker(\overline{\rho}(\sigma_q)-1_m)^i$ of k^m defines a closed point of some flag scheme over k. Let the formal completion of this flag scheme at this closed point be $\mathrm{Spf}\, R_\infty$ and let $\{\mathrm{Fil}^i_{\mathrm{univ}} R_\infty^m\}$ denote the universal lifting of $\{\ker(\overline{\rho}(\sigma_q)-1_m)^i\}$ to a filtration by direct summands of R_∞^m. If we set

$$m_i = \dim_k \ker(\overline{\rho}(\sigma_q)-1_m)^i / \ker(\overline{\rho}(\sigma_q)-1_m)^{i-1}$$

then

$$R_\infty \cong \mathcal{O}[[X_1, ..., X_{(m(m-1)-\sum_i m_i(m_i-1))/2}]].$$

Also let $P \subset GL_m/R_\infty$ be the parabolic subgroup consisting of elements which stabilise $\{\mathrm{Fil}^i_{\mathrm{univ}}\}$. Note that $\overline{\rho} : T_q \to P(k)$. Also note that we have a natural map

$$R_\infty \longrightarrow R_{\overline{\rho}}^{\min}$$

determined by $\{\ker(\rho(\sigma_q)-1_m)^i\}$.

For i a positive integer let P_i the subgroup of $GL_{m_{i+1}+m_{i+2}+\ldots}/R_\infty$ which preserves the filtration $\{\mathrm{Fil}^j_{\mathrm{univ}}/\mathrm{Fil}^i_{\mathrm{univ}}\}$ of $R_\infty^m/\mathrm{Fil}^i_{\mathrm{univ}} R_\infty^m$. Thus $P_0 = P$ and there are natural maps $P_i \to P_{i+1}$. Let $\overline{\rho}_i$ denote the composite

$$T_q \xrightarrow{\overline{\rho}} P(k) \twoheadrightarrow P_i(k).$$

Consider the following functor from Artinian local R_∞-algebras to sets. It sends $R_\infty \to R$ to the set of continuous homomorphisms $\rho_i : T_q \to P_i(R)$ which lift $\overline{\rho}_i : T_q \to P_i(k)$ and for which $\rho_i(\sigma_q)$ acts trivially on each $\mathrm{gr}^j_{\mathrm{univ}} R^m$ for $j > i$. We shall call such lifts ρ_i minimally ramified. This functor is represented by

$$\rho_i^{\mathrm{univ}} : T_q \longrightarrow P_i(R_i),$$

for some complete noetherian local R_∞-algebra R_i. There are natural maps

$$R_i \longrightarrow R_{i-1}.$$

Moreover $R_0 \xrightarrow{\sim} R_{\overline{\rho}}^{\min}$ and for $i >> 0$ we have $R_\infty \xrightarrow{\sim} R_i$.

It suffices to prove that for all i the ring R_{i-1} is a power series ring over R_i in $m_i(m_i+m_{i+1}+...)$ variables. Write

$$\rho_{i-1}^{\mathrm{univ}}(\sigma_q)=\begin{pmatrix}1_{m_i} & X\\ 0 & \rho_i^{\mathrm{univ}}(\sigma_q)\end{pmatrix} \qquad \rho_{i-1}^{\mathrm{univ}}(\phi_q)=\begin{pmatrix}A & B\\ 0 & \rho_i^{\mathrm{univ}}(\phi_q)\end{pmatrix}.$$

We require only one relation

$$(A\ \ B)\begin{pmatrix}X\\ \rho_i^{\mathrm{univ}}(\sigma_q)-1\end{pmatrix}=X(1+\rho_i^{\mathrm{univ}}(\sigma_q)+...+\rho_i^{\mathrm{univ}}(\sigma_q)^{q-1})\rho_i^{\mathrm{univ}}(\phi_q).$$

The reduction modulo $\mathfrak{m}_{R_{i-1}}$ of the matrix

$$Y=\begin{pmatrix}X\\ \rho_i^{\mathrm{univ}}(\sigma_q)-1\end{pmatrix}$$

has the same rank as $\overline{\rho}_{i-1}(\sigma_q)-1$, which is $m_{i+1}+m_{i+2}+....$ Choose $m_{i+1}+m_{i+2}+...$ linearly independent rows of $Y \bmod \mathfrak{m}_{R_{i-1}}$. Then the liftings of X and the m_i columns of $(A\ B)$ not corresponding to the selected rows of $Y \bmod \mathfrak{m}_{R_{i-1}}$ are arbitrary, and the liftings of the remaining columns of $(A\ B)$ are then completely determined. Thus R_{i-1} is indeed a power series ring in

$$m_i(m_{i+1}+m_{i+2}+...)+m_i^2$$

variables over R_i, and the lemma follows. $\Box$

Corollary 2.4.20 *Keep the notation and assumptions of the lemma. Minimally ramified liftings are liftable. Moreover*

$$\dim_k L_v(\mathcal{D}_v^T)=\dim_k H^0(T_q,\mathrm{ad}\,\overline{\rho}).$$

(See definition 2.2.4 for the definition of $L_v(\mathcal{D}_v^T)\subset H^1(T_q,\mathrm{ad}\,\overline{\rho})$.)

Proof: The first assertion is immediate. The second follows from the discussion immediately following definition 2.2.4. $\Box$

Corollary 2.4.21 *Suppose that $\overline{r}: G_{F_{\tilde{v}}}\to GL_n(k)$ is a continuous representation. Define a local deformation problem $\mathcal{D}_v$ to consist of all minimally ramified lifts of $\overline{r}$.*

1. $\mathcal{D}_v$ is liftable.

2. The space L_v of deformations of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ has dimension equal to $\dim_k H^0(G_{F_{\tilde{v}}},\mathrm{ad}\,\overline{r})$.

3. The corresponding quotient $R_v^{\mathrm{loc}}/\mathcal{I}_v$ is a power series ring in n^2 variables over $\mathcal{O}$.

Proof: The first two parts follow from the previous corollary using the equivalence of categories of corollary 2.4.13 and the equality

$$\dim_k H^0(G_{F_{\widetilde{v}}}, \operatorname{ad} \overline{r}) = \sum_{[\tau]} \dim_k H^0(T_\tau, \operatorname{ad} \overline{r}_\tau)$$

(see lemma 2.4.12). The third part follows from the first two. □

Lemma 2.4.22 *Suppose that $l \not| \# \overline{r}(I_{F_{\widetilde{v}}})$ and that $\mathcal{D}_v$ consists of all minimal lifts of $\overline{r}$. Then $L_v = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\operatorname{ad} \overline{r})^{I_{F_{\widetilde{v}}}})$.*

Proof: A lifting of $\overline{r}$ is minimal if and only if it vanishes on $\ker \overline{r}|_{I_{F_{\widetilde{v}}}}$. Thus

$$L_v = H^1(G_{F_{\widetilde{v}}}/(\ker \overline{r}|_{I_{F_{\widetilde{v}}}}), \operatorname{ad} \overline{r}).$$

However $H^1(\overline{r}(I_{F_{\widetilde{v}}}), \operatorname{ad} \overline{r}) = (0)$ so that

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\operatorname{ad} \overline{r})^{I_{F_{\widetilde{v}}}}) \xrightarrow{\sim} H^1(G_{F_{\widetilde{v}}}/(\ker \overline{r}|_{I_{F_{\widetilde{v}}}}), \operatorname{ad} \overline{r})$$

and the lemma follows. □

2.4.5. *Discrete series deformations.* — Let $n = md$ be a factorisation and let

$$\widetilde{r}_v : G_{F_{\widetilde{v}}} \longrightarrow GL_d(\mathcal{O})$$

be a continuous representation such that

1. $\widetilde{r}_v \otimes k$ is absolutely irreducible,
2. every irreducible subquotient of $(\widetilde{r}_v \otimes k)|_{I_{F_{\widetilde{v}}}}$ is absolutely irreducible,
3. and $\widetilde{r}_v \otimes k \not\cong \widetilde{r}_v \otimes k(i)$ for $i = 1, ..., m$.

The second condition is probably unnecessary, but it is harmless for applications and simplifies this section, so we include it. Note that in particular we have

$$k(i) \not\cong k$$

for $i = 1, ..., m$.

Lemma 2.4.23 *1. There is a factorisation $d = d_1 d_2$ and a representation*

$$s_v : G_{F'_{\widetilde{v}}} \longrightarrow GL_{d_2}(\mathcal{O}),$$

where $F'_{\widetilde{v}}/F_{\widetilde{v}}$ is the unramified extension of degree d_1, such that $s_v|_{I_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} k$ is absolutely irreducible and not isomorphic to its conjugate by any element of $G_{F_{\widetilde{v}}} - G_{F'_{\widetilde{v}}}$, and such that

$$\widetilde{r}_v \cong \operatorname{Ind}_{G_{F'_{\widetilde{v}}}}^{G_{F_{\widetilde{v}}}} s_v.$$

2. If R is an object of $\mathcal{C}^f_{\mathcal{O}}$ and $\rho : G_{F_{\widetilde{v}}} \to GL_d(R)$ satisfies

$$\rho|_{I_{F_{\widetilde{v}}}} \cong \widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} R$$

then

$$\rho \cong \mathrm{Ind}\,^{G_{F_{\widetilde{v}}}}_{G_{F'_{\widetilde{v}}}}(s_v \otimes_{\mathcal{O}} R(\chi))$$

for some uniquely determined unramified character $\chi : G_{F'_{\widetilde{v}}} \to R^\times$. In particular

$$\rho \not\cong \rho(i)$$

for $i = 1, ..., m$.

3. If R is an object of $\mathcal{C}^f_{\mathcal{O}}$ and I is an ideal of R then

$$Z_{GL_d(R)}(\widetilde{r}_v(I_{F_{\widetilde{v}}})) \twoheadrightarrow Z_{GL_d(R/I)}(\widetilde{r}_v(I_{F_{\widetilde{v}}})).$$

Proof: Let $\overline{r}_1$ be an irreducible (and hence absolutely irreducible) subrepresentation of $\widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \otimes k$. Let $H \subset G_{F_{\widetilde{v}}}$ denote the group of $\sigma \in G_{F_{\widetilde{v}}}$ such that $\overline{r}_1^\sigma \cong \overline{r}_1$. Because $H/I_{F_{\widetilde{v}}}$ is pro-cyclic we can extend $\overline{r}_1$ to a representation of H. Then there is an H-equivariant embedding

$$\overline{r}_1 \otimes \mathrm{Hom}\,_{I_{F_{\widetilde{v}}}}(\overline{r}_1, \widetilde{r}_v \otimes_{\mathcal{O}} k) \hookrightarrow \widetilde{r}_v \otimes_{\mathcal{O}} k,$$

and the image is the biggest $I_{F_{\widetilde{v}}}$-submodule of $\widetilde{r}_v \otimes_{\mathcal{O}} k$ isomorphic to a direct sum of copies of $\overline{r}_1$. Because $\widetilde{r}_v \otimes_{\mathcal{O}} k$ is absolutely irreducible we see that the map

$$\mathrm{Ind}\,^{G_{F_{\widetilde{v}}}}_H(\overline{r}_1 \otimes \mathrm{Hom}\,_{I_{F_{\widetilde{v}}}}(\overline{r}_1, \widetilde{r}_v \otimes_{\mathcal{O}} k)) \longrightarrow \widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \otimes k$$

is an isomorphism and that $\mathrm{Hom}\,_{I_{F_{\widetilde{v}}}}(\overline{r}_1, \widetilde{r}_v \otimes_{\mathcal{O}} k)$ is an absolutely irreducible $H/I_{F_{\widetilde{v}}}$-module, which must therefore be one dimensional. Twisting $\overline{r}_1$ by a character of $H/I_{F_{\widetilde{v}}}$ we may assume that

$$\widetilde{r}_v \otimes k = \mathrm{Ind}\,^{G_{F_{\widetilde{v}}}}_H \overline{r}_1$$

where $\overline{r}_1|_{I_{F_{\widetilde{v}}}}$ is absolutely irreducible. Thus

$$\widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \otimes k = \overline{r}_1 \oplus ... \oplus \overline{r}_{d_1}$$

where each $\overline{r}_i$ is irreducible, where $\overline{r}_i \not\cong \overline{r}_j$ if $i \neq j$, and where $d_1 = [G_{F_{\widetilde{v}}} : H]$ and $d_1 \dim_k \overline{r}_1 = d$. Note that H is nothing else than $G_{F'_{\widetilde{v}}}$.

We claim that $\widetilde{r}_v|_{I_{F_{\widetilde{v}}}} = r_1 \oplus ... \oplus r_{d_1}$ where r_i is a lifting of $\overline{r}_i$. We prove this modulo λ^t by induction on t, the case $t = 1$ being immediate. So suppose this is true modulo λ^t. As $I_{F_{\widetilde{v}}}$ has cohomological dimension 1 we see that we may lift r_i to a continuous representation $r'_i : I_{F_{\widetilde{v}}} \to GL_{\dim \overline{r}_1}(\mathcal{O}/\lambda^{t+1})$. Then $\widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \bmod \lambda^{t+1}$ differs from $r'_1 \oplus ... \oplus r'_{d_1}$ by an element of

$$H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, \widetilde{r}_v \otimes k) = \bigoplus_{i,j} H^1(I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{r}_i, \overline{r}_j)).$$

For $i \neq j$ we have $\mathrm{Hom}\,(\overline{r}_i, \overline{r}_j)_{I_{F_{\widetilde{v}}}} = (0)$ so

$$H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, \widetilde{r}_v \otimes k) = \bigoplus_{i} H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}_i).$$

Hence $\widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \bmod \lambda^{t+1} = r_1 \oplus ... \oplus r_{d_1}$, as desired.

The group H must stabilise the subspace r_1 and so we can extend r_1 to a representation s_v of H which embeds into $\widetilde{r}_v|_H$ and lifts $\overline{r}_1$. The first part of the lemma follows.

For the second part we are assuming that we have a decomposition

$$\rho|_{I_{F_{\widetilde{v}}}} \cong (r_1 \otimes_{\mathcal{O}} R) \oplus ... \oplus (r_{d_1} \otimes_{\mathcal{O}} R).$$

The submodule $r_1 \otimes_{\mathcal{O}} R$ of ρ is stable by H and so we can extend $r_1 \otimes_{\mathcal{O}} R$ to a representation ρ_1 of H which embeds into $\rho|_H$. We see that

$$\mathrm{Ind}\,_H^{G_{F_{\widetilde{v}}}} \rho_1 \xrightarrow{\sim} \rho.$$

Let ϕ_H denote the lift to H of a topological generator of the pro-cyclic group $H/I_{F_{\widetilde{v}}}$. As $\overline{r}_1|_{I_{F_{\widetilde{v}}}}$ is absolutely irreducible, it follows from lemma 2.1.8 that $(s_v \otimes_{\mathcal{O}} R)(\phi_H)$ and $\rho_1(\phi_H)$ differ by multiplication by an element of $R^\times$, i.e. that $\rho_1 \cong s_v \otimes_{\mathcal{O}} R(\chi)$ for some character $\chi : H/I_{F_{\widetilde{v}}} \to R^\times$.

If

$$\mathrm{Ind}\,_{G_{F'_{\widetilde{v}}}}^{G_{F_{\widetilde{v}}}} (s_v \otimes_{\mathcal{O}} R(\chi)) \cong \mathrm{Ind}\,_{G_{F'_{\widetilde{v}}}}^{G_{F_{\widetilde{v}}}} (s_v \otimes_{\mathcal{O}} R(\chi'))$$

for two characters $\chi, \chi' : H/I_{F_{\widetilde{v}}} \to R^\times$ then

$$s_v \otimes_{\mathcal{O}} R(\chi) \cong s_v \otimes_{\mathcal{O}} R(\chi').$$

But

$$\mathrm{Hom}\,_H(s_v \otimes_{\mathcal{O}} R(\chi), s_v \otimes_{\mathcal{O}} R(\chi')) \cong \mathrm{Hom}\,_{I_{F_{\widetilde{v}}}}(s_v, s_v)(\chi'\chi^{-1})^H \cong R(\chi'\chi^{-1})^H$$

by lemma 2.1.8. Thus we see that $\chi = \chi'$, and the second part of the lemma follows.

For the third part simply note that by lemma 2.1.8

$$Z_{GL_d(R)}(\widetilde{r}_v(I_{F_{\widetilde{v}}})) = (R^\times)^{d_1}.$$

□

Definition 2.4.24 *Suppose that R is an object $\mathcal{C}^f_{\mathcal{O}}$ and*

$$\rho : G_{F_{\widetilde{v}}} \longrightarrow GL_n(R)$$

*is a continuous representation. We will say that ρ is $\widetilde{r}_v$-*discrete series *if there is a decreasing filtration $\{\mathrm{Fil}^i\}$ of ρ by R-direct summands such that*

1. $\mathrm{gr}^i\rho \cong (\mathrm{gr}^0\rho)(i)$ *for $i = 1, ..., m-1$, and*
2. $(\mathrm{gr}^0\rho)|_{I_{F_{\widetilde{v}}}} \cong (\widetilde{r}_v|_{I_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} R)$*.*

Lemma 2.4.25 *If ρ is $\widetilde{r}_v$-discrete series then the filtration $\{\mathrm{Fil}^i\}$ as in the definition is unique.*

Proof: Suppose that $\{\mathrm{Fil}^i_1\}$ and $\{\mathrm{Fil}^i_2\}$ are two such filtrations. Suppose also that

$$\mathrm{gr}^0_j\rho \cong \mathrm{Ind}^{G_{F_{\widetilde{v}}}}_{G_{F'_{\widetilde{v}}}}(s_v \otimes \chi_j).$$

From our assumptions on $\widetilde{r}_v$ we see that

$$\epsilon|^i_{G_{F'_{\widetilde{v}}}} \bmod \lambda \neq 1$$

for $i = 1, ..., m$. However

$$\{\chi_1\epsilon^i \bmod \mathfrak{m}_R\}_{i=0,...,m-1} = \{\chi_2\epsilon^i \bmod \mathfrak{m}_R\}_{i=0,...,m-1}.$$

Thus

$$\chi_2 \equiv \chi_1\epsilon^{i_0} \bmod \mathfrak{m}_R$$

for some $0 \leq i_0 < m$. If $i_0 > 1$ then

$$\chi_1\epsilon^m \cong \chi_2\epsilon^{m-i_0} \cong \chi_1\epsilon^{i_1} \bmod \mathfrak{m}_R$$

for some $0 \leq i_1 < m$, which would give a contradiction. Thus $\chi_1 \equiv \chi_2 \bmod \mathfrak{m}_R$ and

$$\mathrm{gr}^0_1\rho \otimes_R k \cong \mathrm{gr}^0_1\rho \otimes_R k.$$

Note that $\mathrm{gr}^i_j\rho$ is the maximal submodule of $\rho/\mathrm{Fil}^{i+1}_j\rho$ all whose simple $R[G_{F_{\widetilde{v}}}]$-subquotients are isomorphic to $\mathrm{gr}^0_j\rho \otimes k(\epsilon^i)$. Thus by reverse induction on i we see that $\mathrm{Fil}^i_1\rho = \mathrm{Fil}^i_2\rho$. $\Box$

Lemma 2.4.26 *If $\overline{r}$ is $\widetilde{r}_v$-discrete series then the set $\mathcal{D}_v$ of $\widetilde{r}_v$ discrete series liftings of $\overline{r}$ form a local deformation problem.*

Proof: The first two conditions of definition 2.2.2 are immediate. The third and fourth follow from lemma 2.1.8, the third part of lemma 2.4.23 and lemma 2.4.25. The fifth condition is also immediate. Let us verify the sixth condition. Suppose that $R \hookrightarrow S$ is an injective morphism in $\mathcal{C}^f_{\mathcal{O}}$ and that $\rho : G_{F_{\widetilde{v}}} \to GL_n(R)$ is a continuous representation such that ρ thought of as valued in $GL_n(S)$ is $\widetilde{r}_v$-discrete series. Let $\{\mathrm{Fil}^i_S\}$ be the corresponding filtration of S^n and set $\mathrm{Fil}^i_R = \mathrm{Fil}^i_S \cap R^n$. Note that all simple $R[G_{F_{\widetilde{v}}}]$ subquotients of gr^i_S are isomorphic to $\mathrm{gr}^0_S \otimes_S k(i)$. Thus the same is true for all simple subquotients of gr^i_R and hence for $\langle \mathrm{Fil}^i_R \rangle_S / \langle \mathrm{Fil}^{i+1}_R \rangle_S$. By part two of lemma 2.4.23 we see that the $\mathrm{gr}^0_S \otimes_S k(i)$ are non-isomorphic for $i = 0, ..., m-1$ and hence $\langle \mathrm{Fil}^i_R \rangle_S = \mathrm{Fil}^i_S$. In particular the reduction map gives a surjection $\mathrm{Fil}^i_R \twoheadrightarrow \mathrm{Fil}^i_S \otimes_S k \subset k^n$. Choose a basis $\overline{e}_1, ..., \overline{e}_n$ of k^n adapted to $\{\mathrm{Fil}^i_S \otimes_S k\}$. We now see that we can lift it to a basis $e_1, ..., e_n$ of R^n so that $e_i \in \mathrm{Fil}^j_R$ whenever $\overline{e}_i \in \mathrm{Fil}^i_S \otimes_S k$. Then each Fil^j_S has a basis consisting of a subset of the $\{e_i\}$, so that the same is true of Fil^j_R. Thus each Fil^j_R is a direct summand of R^n and $\mathrm{gr}^j_R \otimes_R S \xrightarrow{\sim} \mathrm{gr}^j_S$. The sixth condition of definition 2.2.2 now follows from lemma 2.1.9, lemma 2.1.8 and the third part of lemma 2.4.23. $\square$

For the rest of this section we will assume that $\overline{r}$ is $\widetilde{r}_v$-discrete series and let $\mathcal{D}_v$ denote the set of $\widetilde{r}_v$-discrete series lifts.

Lemma 2.4.27 *$\mathcal{D}_v$ is liftable.*

Proof: We will argue by induction on m. The result for $m = 1$ follows from part 2 of lemma 2.4.23.

Let R be an object of $\mathcal{C}^f_{\mathcal{O}}$ and let I be an ideal of R with $\mathfrak{m}_R I = (0)$. Suppose that r is a $\widetilde{r}_v$-discrete series lifting of $\overline{r}$ to R/I. Let $\{\mathrm{Fil}^i\}$ be the corresponding filtration of r. By the inductive hypothesis we may choose a $\widetilde{r}_v$-discrete series lifting r' of $r/\mathrm{Fil}^{m-1} r$ to R. It will suffice to show that the natural map

$$\begin{array}{c} H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(r', (\mathrm{gr}^0 r')(m-1))) \\ \downarrow \\ H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(r/\mathrm{Fil}^{m-1} r, (\mathrm{gr}^0 r)(m-1))) \end{array}$$

is surjective. The cokernel of this map equals the kernel of

$$\begin{array}{c} H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, (\mathrm{gr}^0 \overline{r})(m-1))) \otimes_k I \\ \downarrow \\ H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(r', (\mathrm{gr}^0 r')(m-1))). \end{array}$$

Using local duality we see that it will suffice to show that

$$\begin{array}{c} H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{gr}^0 r', r'/\mathrm{Fil}^{m-1} r')(2-m)) \otimes_R R^\vee \\ \downarrow \\ H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{gr}^0 \overline{r}, \overline{r}/\mathrm{Fil}^{m-1} \overline{r})(2-m)) \otimes_k I^\vee \end{array}$$

is surjective, where $M^\vee$ denotes the Pontriagin dual of M. However the composites

$$\begin{aligned} k &\xrightarrow{\sim} H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{gr}^0\overline{r}, \mathrm{gr}^{m-2}\overline{r})(2-m)) \\ &\hookrightarrow H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{gr}^0\overline{r}, \overline{r}/\mathrm{Fil}^{m-1}\overline{r})(2-m)) \end{aligned}$$

and

$$\begin{aligned} R &\xrightarrow{\sim} H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{gr}^0 r', \mathrm{gr}^{m-2} r')(2-m)) \\ &\hookrightarrow H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{gr}^0 r', r'/\mathrm{Fil}^{m-1} r')(2-m)) \end{aligned}$$

are isomorphisms, because

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{gr}^0 r', \mathrm{gr}^i r')(2-m)) = (0)$$

and

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{gr}^0\overline{r}, \mathrm{gr}^i\overline{r})(2-m)) = (0)$$

for $i = 0, ..., m-3$. The lemma follows. □

Lemma 2.4.28 $R_v^{\mathrm{loc}}/\mathcal{I}_v$ *is a power series ring in* n^2 *variables over* $\mathcal{O}$.

Proof: We will prove by induction on m that the dimension of the space of $\widetilde{r}_v$-discrete series liftings of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ is n^2. The lemma will follow because $\widetilde{r}_v$-discrete series lifts are liftable.

If $m = 1$ then it follows from part 2 of lemma 2.4.23 that the space of $\widetilde{r}_v$-discrete series *deformations* of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ has dimension 1. Thus the space of $\widetilde{r}_v$-discrete series liftings has dimension:

$$1 + n^2 - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = n^2.$$

Now suppose that $m > 1$. To choose an $\widetilde{r}_v$-discrete series lifting of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ is equivalent to choosing

- a lift $\widetilde{\mathrm{Fil}}^{m-1}$ of $\mathrm{Fil}^{m-1}\overline{r}$ to $(k[\epsilon]/(\epsilon^2))^n$;
- an $\widetilde{r}_v$-discrete series lift r_1 of $\overline{r}/\mathrm{Fil}^{m-1}\overline{r}$ to $k[\epsilon]/(\epsilon^2)$;
- a lifting r_2 of $\mathrm{Fil}^{m-1}\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ such that $r_2 \cong \mathrm{gr}^0 r_1(m-1)$;
- an element of a specific fibre of

$$Z^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_{k[\epsilon]/(\epsilon^2)}(r_1, r_2)) \longrightarrow Z^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r})).$$

The space of choices for $\widetilde{\mathrm{Fil}}^{m-1}$ has dimension $m(n-m)$. The space of choices for r_1 has dimension $(n-m)^2$ by inductive hypothesis. The space of choices for r_2 then has dimension

$$m^2 - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\mathrm{gr}^0\overline{r}) = m^2 - 1.$$

Finally as in the proof of the last lemma, we see that

$$Z^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_{k[\epsilon]/(\epsilon^2)}(r_1, r_2)) \longrightarrow Z^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r}))$$

is surjective with kernel $Z^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r}))$. Thus any fibre has dimension

$$\begin{aligned}
&\dim_k Z^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r})) \\
&= m(n-m) - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r})) \\
&\quad + \dim_k H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r})) \\
&= m(n-m) + \dim_k H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\overline{r}/\mathrm{Fil}^{m-1}\overline{r}, \mathrm{Fil}^{m-1}\overline{r})) \\
&= m(n-m) + \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{Fil}^{m-1}\overline{r}, \overline{r}/\mathrm{Fil}^{m-1}\overline{r})(1)).
\end{aligned}$$

(We are using the exact sequence in the paragraph following definition 2.2.1, the local Euler characteristic formula and local duality.) As in the proof of the last lemma we see that

$$k \cong H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathrm{Fil}^{m-1}\overline{r}, \overline{r}/\mathrm{Fil}^{m-1}\overline{r})(1)).$$

Thus the space of $\widetilde{r}_v$-discrete series liftings of $\overline{r}$ to $k[\epsilon]/(\epsilon^2)$ has dimension

$$m(n-m) + (n-m)^2 + (m^2-1) + (m(n-m)+1) = n^2.$$

The lemma follows. $\square$

Corollary 2.4.29 *Keep the notation of the lemma. Then*

$$\dim_k L_v = \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}).$$

The next lemma is self-explanatory.

Lemma 2.4.30 *Suppose that $d=1$ and $m=n$. Define $\mathrm{Fil}^1\mathrm{ad}\,\overline{r}$ to be the set of x in $\mathrm{ad}\,\overline{r}$ such that $x\mathrm{Fil}^i\overline{r} \subset \mathrm{Fil}^{i+1}\overline{r}$ for all i. If $\mathcal{D}_v$ is the set of discrete series lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ then*

$$L_v = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n) \oplus \ker(H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}^0\overline{r}) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^1\mathrm{ad}\,\overline{r}).$$

2.4.6. *Taylor-Wiles deformations.* — Suppose that $\mathbf{N}\widetilde{v} \equiv 1 \bmod l$, that $\overline{r}$ is unramified at $\widetilde{v}$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi}_v \oplus \overline{s}_v$ where $\dim_k \overline{\psi}_v = 1$ and $\overline{s}_v$ does not contain $\overline{\psi}_v$ as a sub-quotient. Take $\mathcal{D}_v$ to consist of all lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1+M_n(\mathfrak{m}_R))$-conjugate to one of the form $\psi \oplus s$ where ψ lifts $\overline{\psi}_v$, and where s lifts $\overline{s}_v$ and is unramified. Then $\mathcal{D}_v$ is a local deformation problem and

$$L_v = L_v(\mathcal{D}_v) = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}_v) \oplus H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{\psi}_v).$$

Note that in this case

$$\mathrm{lg}_{\mathcal{O}} L_v - \mathrm{lg}_{\mathcal{O}} H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = \mathrm{lg}_{\mathcal{O}} H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{\psi}_v)^{G_{F_{\widetilde{v}}}} = 1.$$

We will write Δ_v for the maximal l-power quotient of the inertia subgroup of $G_{F_{\widetilde{v}}}^{\mathrm{ab}}$. It is cyclic of order the maximal power of l dividing $\mathbf{N}\widetilde{v}-1$. If r is any deformation of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ in $\mathcal{D}_v$ over a ring R then $\det r : \Delta_v \to R^\times$ and so R becomes an $\mathcal{O}[\Delta_v]$-algebra. If $\mathfrak{a}_v$ denotes the augmentation ideal of $\mathcal{O}[\Delta_v]$ then $R/\mathfrak{a}_v R$ is the maximal quotient of R over which r becomes unramified at v.

2.4.7. *Ramakrishna deformations.* — Suppose that $(\mathbf{N}\widetilde{v}) \not\equiv 1 \bmod l$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi}_v\epsilon \oplus \overline{\psi}_v \oplus \overline{s}_v$, where $\overline{\psi}_v$ and $\overline{s}_v$ are unramified and $\overline{s}_v$ contains neither $\overline{\psi}_v$ nor $\overline{\psi}_v\epsilon$ as a sub-quotient. Take $\mathcal{D}_v$ to consist of the set of lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1+M_n(\mathfrak{m}_R))$-conjugate to a lift of the form

$$\begin{pmatrix} \psi\epsilon & * & 0 \\ 0 & \psi & 0 \\ 0 & 0 & s \end{pmatrix}$$

with ψ an unramified lift of $\overline{\psi}_v$ and s an unramified lift of $\overline{s}_v$. Then $\mathcal{D}_v$ is a local deformation problem and $L_v = L_v(\mathcal{D}_v)$ is

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k\begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix}) \oplus H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\psi}_v, \overline{\psi}_v\epsilon)) \oplus H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}_v).$$

Then

$$\begin{aligned} &\dim_k L_v \\ &= 2 + \dim_k H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}_v) \\ &= 2 + \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}_v) \\ &= \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}). \end{aligned}$$

Moreover $\mathcal{D}_v$ is liftable. (Because if R is an object of $\mathcal{C}_{\mathcal{O}}$ and if I is a closed ideal of R then

$$H^1(G_{F_{\widetilde{v}}}, R(\epsilon)) \twoheadrightarrow H^1(G_{F_{\widetilde{v}}}, (R/I)(\epsilon)).)$$

2.4.8. *One more local deformation problem.* — Suppose again that $(\mathbf{N}\widetilde{v}) \not\equiv 1 \bmod l$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi}_v\epsilon \oplus \overline{\psi}_v \oplus \overline{s}_v$, where $\overline{\psi}_v$ and $\overline{s}_v$ are unramified and $\overline{s}_v$ contains neither $\overline{\psi}_v$ nor $\overline{\psi}_v\epsilon$ as a sub-quotient. Take $\mathcal{D}_v$ to consist of the set of lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1+M_n(\mathfrak{m}_R))$-conjugate to a lift of the form

$$\begin{pmatrix} \psi_1 & * & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & s \end{pmatrix}$$

with ψ_1 (resp. ψ_2) an unramified lift of $\overline{\psi}_v\epsilon$ (resp. $\overline{\psi}_v$) and s an unramified lift of $\overline{s}_v$. Note that $\mathcal{D}_v$ includes all unramified lifts and all Ramakrishna lifts (see section 2.4.7). It is a local deformation problem and $L_v = L_v(\mathcal{D}_v)$ is

$$H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, \mathrm{Hom}\,(\overline{\psi}_v\epsilon, \overline{\psi}_v\epsilon) \oplus \mathrm{Hom}\,(\overline{\psi}_v, \overline{\psi}_v)) \oplus H^1(G_{F_{\tilde{v}}}, \mathrm{Hom}\,(\overline{\psi}_v, \overline{\psi}_v\epsilon)) \oplus \\ \oplus H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{s}_v).$$

Then $\dim_k L_v = 3 + \dim_k H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{s}_v) = 3 + \dim_k H^0(G_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{s}_v) = 1 + \dim_k H^0(G_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{r})$.

We remark that this deformation problem is only used in the proof of theorem 2.6.3, where its function is to compare unramified deformations with Ramakrishna deformations.

2.5. *An application of the Cebotarev Density Theorem.* — We will keep the notation and assumptions established at the start of section 2.3. In this section we will lay the groundwork for the Taylor-Wiles arguments we will use to prove our modularity lifting theorems. More specifically we will use the Cebotarev density theorem and our Galois cohomology calculations to construct the sets of auxiliary primes on which the method relies. To be able to do this we will need to put some restrictions on the image of $\overline{r}$. The condition we will need to impose we have called 'big'. This condition is somewhat ugly, but we failed to find a more natural formulation. It is however usually easy to verify in specific cases. The terminology 'big' is perhaps unfortunate. If the cardinality of a subgroup $H \subset \mathcal{G}_n(k)$ is large compared to the cardinality of $\mathcal{G}_n(k)$ then the H is often 'big' in our technical sense. However there are also many subgroups $H \subset \mathcal{G}_n(k)$ whose cardinality is not large which are also 'big' in our technical sense. We apologise for our lack of imagination in nomenclature.

Definition 2.5.1 *We will call a subgroup $H \subset \mathcal{G}_n(k)$* big *if the following conditions are satisfied.*

- *$H \cap \mathcal{G}_n^0(k)$ has no l-power order quotient.*
- *$H^0(H, \mathfrak{g}_n(k)) = (0)$.*
- *$H^1(H, \mathfrak{g}_n(k)) = (0)$.*
- *For all irreducible $k[H]$-submodules W of $\mathfrak{g}_n(k)$ we can find $h \in H \cap \mathcal{G}_n^0(k)$ and $\alpha \in k$ with the following properties. The α generalised eigenspace $V_{h,\alpha}$ of h in k^n is one dimensional. Let $\pi_{h,\alpha} : k^n \to V_{h,\alpha}$ (resp. $i_{h,\alpha}$) denote the h-equivariant projection of k^n to $V_{h,\alpha}$ (resp. h-equivariant injection of $V_{h,\alpha}$ into k^n). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.*

Similarly we call a subgroup $H \subset GL_n(k)$ big *if the following conditions are satisfied.*

– *H has no l-power order quotient.*
– *$H^0(H, \mathfrak{g}_n^0(k)) = (0)$.*
– *$H^1(H, \mathfrak{g}_n^0(k)) = (0)$.*
– *For all irreducible $k[H]$-submodules W of $\mathfrak{g}_n^0(k)$ we can find $h \in H$ and $\alpha \in k$ with the following properties. The α generalised eigenspace $V_{h,\alpha}$ of h in k^n is one dimensional. Let $\pi_{h,\alpha} : k^n \to V_{h,\alpha}$ (resp. $i_{h,\alpha}$) denote the h-equivariant projection of k^n to $V_{h,\alpha}$ (resp. h-equivariant injection of $V_{h,\alpha}$ into k^n). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.*

(Recall that $\mathfrak{g}_n^0$ denotes the trace zero subspace of $\mathrm{Lie}\, GL_n \subset \mathrm{Lie}\, \mathcal{G}_n$.)

We note that the fourth property will also hold for any non-zero $\mathbf{F}_l[H]$-subspace W of $\mathfrak{g}_n(k)$. (Because it holds for W if and only if it holds for its k-linear span.) Also note that, if $H \subset \mathcal{G}_n(k)$ surjects onto $\mathcal{G}_n(k)/\mathcal{G}_n^0(k)$ and if $H \cap \mathcal{G}_n^0(k)$ is big, then H is big.

At the referee's suggestion, we will digress here to give some examples of big subgroups $H \subset \mathcal{G}_n(k)$, which will be needed later.

Lemma 2.5.2 *Suppose that $l > 2n - 1$ is a prime; that k is an algebraic extension of $\mathbf{F}_l$; and that $H \subset GL_n(k)$. Suppose that*

– *H has no l-power order quotient,*
– *H contains $\mathrm{Symm}^{n-1} SL_2(\mathbf{F}_l)$, and*
– *$H^1(H, \mathfrak{g}_n^0(k)) = (0)$.*

Then H is big.

Proof: As a $SL_2(\mathbf{F}_l)$-module we have

$$\mathrm{ad}\, \mathrm{Symm}^{n-1} \cong 1 \oplus \mathrm{Symm}^2 \oplus \mathrm{Symm}^4 \oplus ... \oplus \mathrm{Symm}^{2n-2}.$$

(That $\mathrm{ad}\, \mathrm{Symm}^{n-1}$ is semi-simple follows for instance from [Se2].) As $2n-2 \leq l-1$ each factor in this decomposition is irreducible. In particular

$$H^0(H, \mathfrak{g}_n^0(k)) = (0).$$

Let T denote the torus of diagonal elements in $SL_2(\mathbf{F}_l)$ and let t denote a generator of T. Let $D = (\mathrm{ad}\, \overline{r})^T$. As $n < l$ we can decompose

$$\mathrm{Symm}^{n-1}|_T = V_0 \oplus V_1 \oplus ... \oplus V_{n-1}$$

where the V_i are the eigenspaces of t and each is one dimensional. Let $i_{t,j}$ denote the injection $V_j \hookrightarrow \mathrm{Symm}^{n-1}$ and $\pi_{t,j}$ denote the t-equivariant projection $\mathrm{Symm}^{n-1} \twoheadrightarrow V_j$. Thus $\pi_{t,j} i_{t,j} = 1$. As $2n < l + 1$ we see that

$$D = \bigoplus_{j=0}^{n-1} \mathrm{Hom}\,(V_j, V_j)$$

has dimension n and that, for $i = 0, ..., n-1$

$$\dim D \cap \mathrm{Symm}^{2i} = 1.$$

For each $i = 0, ..., n-1$ choose j such that the projection of $D \cap \mathrm{Symm}^{2i}$ onto $\mathrm{Hom}(V_j, V_j)$ is non-trivial. Then

$$\pi_{t,j}(D \cap \mathrm{Symm}^{2i})i_{t,j} \neq (0).$$

□

Corollary 2.5.3 *Fix positive integers m and n. There is a constant $C(mn^2)$ such that for any prime $l > C(mn^2)$ and any finite extension $k/\mathbf{F}_l$ of degree at most m the group $GL_n(k)$ has the following property. Any subgroup $H \subset GL_n(k)$ which contains $\mathrm{Symm}^{n-1} SL_2(\mathbf{F}_l)$, but has no l-power order quotient, is big.*

Proof: Using the lemma one just needs to check that $H^1(H, \mathfrak{g}_n^0(k)) = (0)$. However [Se2] tells us that $\mathfrak{g}_n(k)$ is semi-simple as an H-module. The result then follows from theorem E of [N]. □

Corollary 2.5.4 *Suppose that $l > 2n-1$ is a prime; that k is an algebraic extension of $\mathbf{F}_l$; that $k' \subset k$ is a finite field and that $H \subset GL_n(k)$. Suppose that*

$$k^\times \mathrm{Symm}^{n-1} GL_2(k') \supset H \supset \mathrm{Symm}^{n-1} SL_2(k').$$

Then H is big.

Proof: It follows from the lemma that it suffices to show that

$$H^1(SL_2(k'), \mathrm{ad}\, \mathrm{Symm}^{n-1}) = (0).$$

(Note that $l \not| [H : \mathrm{Symm}^{n-1} SL_2(k')]$.) As in the proof of the lemma we have a decomposition

$$\mathrm{ad}\, \mathrm{Symm}^{n-1} \cong 1 \oplus \mathrm{Symm}^2 \oplus \mathrm{Symm}^4 \oplus ... \oplus \mathrm{Symm}^{2n-2}.$$

Let B (resp. T) denote the subgroup of $SL_2(k')$ consisting of upper triangular (resp. diagonal) matrices and let U denote the Sylow l-subgroup of B. Thus

$$H^1(SL_2(k'), \mathrm{ad}\, \mathrm{Symm}^{n-1}) \hookrightarrow \bigoplus_{i=0}^{n-1} H^1(U, \mathrm{Symm}^{2i})^B.$$

As $l > n+1$ it follows from lemma (2.7) c) of [CPS] that for $i = 0, ..., n-1$ we have

$$H^1(U, \mathrm{Symm}^{2i})^B = (0).$$

The lemma follows. □

Lemma 2.5.5 *Suppose that n is even; that $l > \max\{3, n\}$ is a prime; that k is an algebraic extension of $\mathbf{F}_l$; that $k' \subset k$ is a finite field; and that $H \subset GL_n(k)$. Suppose that*

$$k^\times GSp_n(k') \supset H \supset Sp_n(k').$$

Then H is big.

Proof: For definiteness we suppose that Sp_n is defined by the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & 1_{n/2} \\ -1_{n/2} & 0 \end{pmatrix},$$

i.e. $Sp_n = \{g \in GL_n : \; gJ^tg = J\}$. Define H-submodules R_0, R_1 and R_2 of $\mathfrak{g}_n(k)$ as follows. R_0 consists of scalar matrices. R_1 consists of matrices A such that $AJ + J^tA = 0$. Finally R_2 consists of matrices A such that $\mathrm{tr}\, A = 0$ and $AJ - J^tA = 0$. Each is preserved by H^0. As $l > n$ we see that

$$\mathrm{ad} = R_0 \oplus R_1 \oplus R_2$$

and each R_i is an irreducible $Sp_n(\mathbf{F}_l)$-module. (The latter fact is because each R_i is a Weyl module with l-restricted highest weight.) Thus $H^0(H, \mathfrak{g}_n^0(k)) = (0)$.

Choose $\alpha \in \mathbf{F}_l^\times$ with $\alpha^2 \neq 1$ and take h to be the diagonal matrix

$$\mathrm{diag}(\alpha, 1, ..., 1, \alpha^{-1}, 1, ..., 1)$$

in $Sp_n(\mathbf{F}_l)$. If i_α (resp. π_α) denotes the injection of (resp. projection onto) the α eigenspace in k^n then

$$\pi_\alpha R_j i_\alpha \neq (0)$$

for $j = 0$, 1 and 2.

Finally it will suffice to check that

$$H^1(Sp_n(k'), \mathfrak{g}_n(k)) = (0),$$

or simply that $H^1(Sp_n(k'), \mathfrak{g}_n^0(k)) = (0)$. (Because $Sp_n(k')$ has no quotient of l-power order.) Let B_n denote the Borel subgroup of elements of Sp_n of the form

$$\begin{pmatrix} a & b \\ 0 & {}^ta^{-1} \end{pmatrix}$$

with a upper triangular. Then $(\mathrm{ad}\, \overline{r})^{B_n(\mathbf{F}_l)} = R_0$. Also let T_n denote the subgroup of Sp_n consisting of diagonal elements. Identify the character group $X^*(T_n)$ with $\mathbf{Z}^{n/2}$ by

$$(a_1, ..., a_{n/2})\mathrm{diag}(t_1, ..., t_{n/2}, t_1^{-1}, ..., t_{n/2}^{-1}) = t_1^{a_1} ... t_{n/2}^{a_{n/2}}.$$

Corollary 2.9 of [CPS] tells us that $H^1(Sp_n(k'), \mathfrak{g}_n^0(k)) = (0)$. (According to footnote (23) on page 182 of [CPS], because $l > 3$, we may take ψ of corollary 2.9 of [CPS] to consist of $(1, -1, 0, ..., 0)$, $(0, 1, -1, ..., 0)$, ..., $(0, 0, ..., 1, -1)$, and $(0, 0, ..., 0, 2)$. Then that corollary tells us that

$$\dim H^1(Sp_n(k'), \mathfrak{g}_n^0(k)) = 2(n/2 - 1) + 1 - (n - 1) = 0.)$$

□

Lemma 2.5.6 *Suppose that $l > n$ is a prime; that k is an algebraic extension of $\mathbf{F}_l$; that $k' \subset k$ is a finite field; and that $H \subset GL_n(k)$. If $n = 2$ suppose further that $l > 3$ and $\#k' > 5$. Suppose that*

$$k^\times GL_n(k') \supset H \supset SL_n(k').$$

Then H is big.

Proof: As $l > n$ we see that

$$\mathfrak{g}_n(k) = \mathfrak{g}_n^0(k) \oplus k1_n$$

as H-modules and that $\mathfrak{g}_n^0(k)$ is an irreducible $SL_n(\mathbf{F}_l)$-module. We deduce that $H^0(H, \mathfrak{g}_n^0(k)) = (0)$.

Choose $\alpha \in \mathbf{F}_l^\times$ with $\alpha^2 \neq 1$ and take h to be the diagonal matrix

$$\mathrm{diag}(\alpha, \alpha^{-1}, ..., 1)$$

in $SL_n(\mathbf{F}_l)$. If i_α (resp. π_α) denotes the injection of (resp. projection onto) the α eigenspace in k^n then

$$\pi_\alpha \mathfrak{g}_n^0(k) i_\alpha \neq (0)$$

and

$$\pi_\alpha k1_n i_\alpha \neq (0).$$

Finally it will suffice to check that

$$H^1(SL_n(k'), \mathfrak{g}_n(k)) = (0),$$

or simply that $H^1(SL_n(k'), \mathfrak{g}_n^0(k)) = (0)$. (Because $SL_n(k')$ has no quotient of l-power order.) But this follows from table (4.5) of [CPS]. □

These examples are by no means exhaustive. We will discuss another example later (see lemma 2.7.5). We wonder whether in any irreducible compatible system of de Rham λ-adic representations from the absolute Galois group of a number field into $\mathcal{G}_n$ with distinct Hodge-Tate numbers, the image of the corresponding mod λ representation will be big for all but finitely many λ.

We now turn to the Galois theoretic part of the Taylor-Wiles argument in this context.

Definition 2.5.7 *Suppose that*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S})$$

is a global deformation problem and that $T \subset S$*. Let* Q *be a finite set of primes* $v \notin S$ *of* F^+ *which split in* F *and for which*

$$\mathbf{N}v \equiv 1 \bmod l.$$

Let $\widetilde{Q}$ *denote the set consisting of one choice* $\widetilde{v}$ *of a prime of* F *above each element of* Q*. For* $v \in Q$ *suppose also that* $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi}_v \oplus \overline{s}_v$ *where* $\dim_k \overline{\psi} = 1$ *and* $\overline{s}$ *does not contain* $\overline{\psi}$ *as a sub-quotient. Then we define a second global deformation problem*

$$\mathcal{S}(Q) = \mathcal{S}(Q, \{\overline{\psi}_v\}_{v\in Q}) = (F/F^+, S\cup Q, \widetilde{S}\cup\widetilde{Q}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S\cup Q}),$$

where for $v \in Q$ *we take* $\mathcal{D}_{\widetilde{v}}$ *to consist of all lifts of* $\overline{r}|_{G_{F_{\widetilde{v}}}}$ *which are* $(1 + M_n(\mathfrak{m}_R))$*-conjugate to one of the form* $\psi \oplus s$ *where* ψ *lifts* $\overline{\psi}_v$*, and where* s *lifts* $\overline{s}$ *and is unramified. (See section 2.4.6.)*

If $v \in Q$ *then we will write* Δ_v *for the maximal* l*-power order quotient of the inertia subgroup of* $G_{F_{\widetilde{v}}}^{\mathrm{ab}}$*. We will also write*

$$\Delta_Q = \prod_{v\in Q} \Delta_v,$$

and $\mathfrak{a}_{T,Q}$ *for the ideal of* $\mathcal{T}_T[\Delta_Q]$ *generated by the* $X_{v,i,j}$ *(for* $v \in T$ *and* $i, j = 1, ..., n$*) and the* $\delta - 1$ *for* $\delta \in \Delta_Q$*. If* $\overline{r}$ *is Schur we have*

$$R^{\square_T}_{\mathcal{S}(Q)}/\mathfrak{a}_{T,Q} = R^{\mathrm{univ}}_{\mathcal{S}}.$$

The next lemma follows immediately from corollary 2.3.5.

Lemma 2.5.8 *Keep the notation and assumptions of the start of section 2.3. Also suppose that* $\overline{r}$ *is Schur and that for* $v \in S - T$ *we have*

$$\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = \begin{cases} [F_v^+ : \mathbf{Q}_l]n(n-1)/2 & \text{if } v|l \\ 0 & \text{if } v \not| l. \end{cases}$$

Let $(Q, \{\overline{\psi}_v\}_{v\in Q})$ *be as in definition 2.5.7. Then* $R^{\square_T}_{\mathcal{S}(Q,\{\overline{\psi}_v\})}$ *can be topologically generated over* $R^{\mathrm{loc}}_{\mathcal{S},T} = R^{\mathrm{loc}}_{\mathcal{S}(Q),T}$ *by*

$$\begin{aligned}&\dim_k H^1_{\mathcal{L}(Q)^\perp,T}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) + \#Q - \textstyle\sum_{v\in T,\ v|l}[F_v^+ : \mathbf{Q}_l]n(n-1)/2- \\ &- \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) - n\textstyle\sum_{v|\infty}(1+\chi(c_v))/2\end{aligned}$$

elements.

Proposition 2.5.9 *Keep the notation and assumptions of the start of section 2.3. Let $q_0 \in \mathbf{Z}_{\geq 0}$. Suppose that $\overline{r}$ is Schur and that the group $\overline{r}(G_{F^+(\zeta_l)})$ is big. Suppose also that for $v \in S-T$ we have*

$$\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}) = \begin{cases} [F_v^+ : \mathbf{Q}_l]n(n-1)/2 & \text{if } v|l \\ 0 & \text{if } v \not| l. \end{cases}$$

Set q to be the larger of $\dim_k H^1_{\mathcal{L}^\perp, T}(G_{F^+,S}, \mathrm{ad}\, \overline{r}(1))$ and q_0. For any positive integer N we can find $(Q, \widetilde{Q}, \{\overline{\psi}_v\}_{v \in Q})$ as in definition 2.5.7, with the following properties.

- *$\#Q = q \geq q_0$.*
- *If $v \in Q$ then $\mathbf{N}v \equiv 1 \bmod l^N$.*
- *$R^{\square_T}_{\mathcal{S}(Q, \{\overline{\psi}_v\})}$ can be topologically generated over $R^{\mathrm{loc}}_{\mathcal{S},T} = R^{\mathrm{loc}}_{\mathcal{S}(Q),T}$ by*

$$\#Q - \sum_{v \in T,\ v|l} [F_v^+ : \mathbf{Q}_l] n(n-1)/2 - n \sum_{v|\infty} (1 + \chi(c_v))/2$$

elements.

Proof: Suppose that $(Q, \{\overline{\psi}_v\}_{v \in Q})$ is as in definition 2.5.7. We have a left exact sequence

$$\begin{array}{r} (0) \longrightarrow H^1(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(\epsilon)) \longrightarrow H^1(G_{F^+, S \cup Q}, (\mathrm{ad}\, \overline{r})(\epsilon)) \longrightarrow \\ \longrightarrow \bigoplus_{v \in Q} H^1(I_{F_{\widetilde{v}}}, (\mathrm{ad}\, \overline{r})(\epsilon))^{G_{F_{\widetilde{v}}}}. \end{array}$$

As

$$H^1(I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\psi}_v, \overline{s}_v)(\epsilon))^{G_{F_{\widetilde{v}}}} = \mathrm{Hom}\,(\overline{\psi}_v, \overline{s}_v)_{G_{F_{\widetilde{v}}}} = (0)$$

and

$$H^1(I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{s}_v, \overline{\psi}_v)(\epsilon))^{G_{F_{\widetilde{v}}}} = \mathrm{Hom}\,(\overline{s}_v, \overline{\psi}_v)_{G_{F_{\widetilde{v}}}} = (0)$$

we have a left exact sequence

$$\begin{array}{l} (0) \longrightarrow H^1(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(\epsilon)) \longrightarrow H^1(G_{F^+, S \cup Q}, (\mathrm{ad}\, \overline{r})(\epsilon)) \longrightarrow \\ \longrightarrow \bigoplus_{v \in Q} (H^1(I_{F_{\widetilde{v}}}, (\mathrm{ad}\, \overline{s}_v)(\epsilon))^{G_{F_{\widetilde{v}}}} \oplus H^1(I_{F_{\widetilde{v}}}, (\mathrm{ad}\, \overline{\psi}_v)(\epsilon))^{G_{F_{\widetilde{v}}}}), \end{array}$$

and hence a left exact sequence

$$\begin{array}{l} (0) \longrightarrow H^1_{\mathcal{L}(Q)^\perp}(G_{F^+, S \cup Q}, (\mathrm{ad}\, \overline{r})(\epsilon)) \longrightarrow H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(\epsilon)) \longrightarrow \\ \longrightarrow \bigoplus_{v \in Q} H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\, \overline{\psi}_v)(\epsilon)) = \bigoplus_{v \in Q} k. \end{array}$$

The latter map sends the class of a cocycle $\phi \in Z^1(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(\epsilon))$ to

$$(\pi_{\mathrm{Frob}_{\widetilde{v}}, \psi_v(\mathrm{Frob}_{\widetilde{v}})} \circ \phi(\mathrm{Frob}_{\widetilde{v}}) \circ i_{\mathrm{Frob}_{\widetilde{v}}, \psi_v(\mathrm{Frob}_{\widetilde{v}})})_{v \in Q}.$$

(We are using $\pi_{h,\alpha}$ (resp. $i_{h,\alpha}$) to denote the h-equivariant projection onto (resp. injection of) the α eigenspace of h.)

By lemma 2.5.8 it suffices to find a set Q of primes of F^+ disjoint from S with $\#Q \geq q_0$ and such that

– if $v \in Q$ then v splits completely in $F(\zeta_{l^N})$;

– if $v \in Q$ then $\overline{r}(\mathrm{Frob}_v)$ has an eigenvalue $\overline{\psi}_v(\mathrm{Frob}_{\widetilde{v}})$ whose generalised eigenspace has dimension 1;

– $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(\epsilon)) \hookrightarrow \bigoplus_{v\in Q} H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{\psi}_v)(\epsilon))$.

(If necessary we can then shrink Q to a set of cardinality q with the same properties.) By the Cebotarev density theorem it suffices to show that if ϕ is an element of the group $Z^1(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(\epsilon))$ with non-zero image in $H^1(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(\epsilon))$, then we can find $\sigma \in G_{F(\zeta_{l^N})}$ such that

– $\overline{r}(\sigma)$ has an eigenvalue α whose generalised eigenspace has dimension 1;

– $\pi_{\sigma,\alpha} \circ \phi(\sigma) \circ i_{\sigma,\alpha} \neq 0$.

Let $L/F(\zeta_{l^N})$ be the extension cut out by $\mathrm{ad}\,\overline{r}$. If $\sigma' \in G_L$ then $\overline{r}(\sigma'\sigma) \in k^\times \overline{r}(\sigma)$ and $\phi(\sigma'\sigma) = \phi(\sigma') + \phi(\sigma)$. Thus it suffices to find $\sigma \in G_{F(\zeta_{l^N})}$ such that

– $\overline{r}(\sigma)$ has an eigenvalue α whose generalised eigenspace has dimension 1;

– $\pi_{\sigma,\alpha} \circ (\phi(G_L) + \phi(\sigma)) \circ i_{\sigma,\alpha} \neq 0$.

It even suffices to find $\sigma \in \mathrm{Gal}\,(L/F(\zeta_{l^N}))$ such that

– $\overline{r}(\sigma)$ has an eigenvalue α whose generalised eigenspace has dimension 1;

– $\pi_{\sigma,\alpha} \circ \phi(G_L) \circ i_{\sigma,\alpha} \neq 0$.

As $\overline{r}(G_{F^+(\zeta_l)})$ is big, so is $\overline{r}(G_{F^+(\zeta_{l^N})})$. Thus $H^1(\mathrm{Gal}\,(L/F(\zeta_{l^N})), \mathrm{ad}\,\overline{r}) = (0)$. We deduce that $[\phi] \neq 0$ implies that $\phi(G_L) \neq (0)$. Then the existence of a suitable σ follows from our assumptions. □

2.6. *Lifting Galois representations.* — In this section we will prove a generalisation of Ramakrishna's lifting theorem for Galois representations [Ra2]. We keep the notation and assumptions at the start of section 2.3.

Definition 2.6.1 *Suppose that* $\mathrm{ad}\,\overline{r}$ *is a semisimple* $k[G_{F^+}]$*-module. If* $W \subset \mathrm{ad}\,\overline{r}$ *is a* $k[G_{F^+}]$*-submodule we will define*

$$\begin{aligned} &H^1_{\mathcal{S}}(G_{F^+,S}, W) \\ &= H^1(G_{F^+,S}, W) \cap H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \\ &= \ker(H^1(G_{F^+,S}, W) \longrightarrow \textstyle\bigoplus_{\widetilde{v}\in\widetilde{S}} H^1(G_{F_{\widetilde{v}}}, W)/(L_{\widetilde{v}} \cap H^1(G_{F_{\widetilde{v}}}, W))) \end{aligned}$$

and

$$\begin{aligned} &H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W(1)) \\ &= H^1(G_{F^+,S}, W(1)) \cap H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) \\ &= \ker(H^1(G_{F^+,S}, W(1)) \longrightarrow \textstyle\bigoplus_{v\in S} H^1(G_{F_{\widetilde{v}}}, W)/(L_v^\perp \cap H^1(G_{F_{\widetilde{v}}}, W(1)))). \end{aligned}$$

We will call W (resp. $W(1)$) insubstantial *if $H^1_{\mathcal{S}}(G_{F^+,S}, W) = (0)$ (resp. $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W(1)) = (0)$).*

Definition 2.6.2 *Suppose that*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S})$$

is a global deformation problem. Let Q be a finite set of primes $v \notin S$ of F^+ which split in F and for which

$$\mathbf{N}v \not\equiv 1 \bmod l.$$

Also let $\widetilde{Q}$ denote a set consisting of one choice of a prime $\widetilde{v}$ of F above each element v of Q. For $v \in Q$ suppose also that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \bar{t}_v \oplus \overline{s}_v$ with $\bar{t}_v = \overline{\psi}_v \oplus \overline{\psi}_v \epsilon$ where $\dim_k \overline{\psi}_v = 1$ and $\overline{s}_v$ does not contain $\overline{\psi}_v$ or $\overline{\psi}_v\epsilon$ as a sub-quotient. Then we define a global deformation problem

$$\mathcal{S}[Q] = \mathcal{S}[Q, \{\overline{\psi}_v\}_{v\in Q}] = (F/F^+, S\cup Q, \widetilde{S}\cup\widetilde{Q}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S\cup Q}),$$

where for $v \in Q$ we take $\mathcal{D}_v$ to consist of all lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1 + M_n(\mathfrak{m}_R))$-conjugate to one of the form $t \oplus s$ where t is an extension of an unramified lift ψ of $\overline{\psi}_v$ by $\psi\epsilon$, and where s is an unramified lift of $\overline{s}$. (See section 2.4.7.) We also define a second new global deformation

$$\mathcal{S}[Q]' = \mathcal{S}[Q, \{\overline{\psi}_v\}_{v\in Q}]' = (F/F^+, S\cup Q, \widetilde{S}\cup\widetilde{Q}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S} \cup \{\mathcal{D}'_v\}_{v\in Q}),$$

where for $v \in Q$ we take $\mathcal{D}'_v$ to consist of all lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1 + M_n(\mathfrak{m}_R))$-conjugate to one of the form $t \oplus s$ where t is an extension of an unramified lift of $\overline{\psi}_v$ by an unramified lift of $\overline{\psi}_v\epsilon$, and where s is an unramified lift of $\overline{s}$. (See section 2.4.8.)

If $v \in Q$ we will let $\pi_{\overline{\psi}_v}$ (resp. $i_{\overline{\psi}_v}$, resp. $\pi_{\overline{\psi}_v\epsilon}$, resp. $i_{\overline{\psi}_v\epsilon}$) denote the $G_{F_{\widetilde{v}}}$-equivariant projection $\overline{r} \twoheadrightarrow \overline{\psi}_v$ (resp. inclusion $\overline{\psi}_v \hookrightarrow \overline{r}$, resp. projection $\overline{r} \twoheadrightarrow \overline{\psi}_v\epsilon$, resp. inclusion $\overline{\psi}_v\epsilon \hookrightarrow \overline{r}$).

We now state our main lifting theorem for Galois representations. We believe such theorems have some intrinsic interest. In addition we will need to apply this theorem in the following situation. We will have a mod l representation which is induced from a character (and hence provably automorphic). We will need to find an l-adic lift whose restriction to the decomposition group at some prime corresponds (under the local Langlands correspondence) to a Steinberg representation. (Such a lift will never itself be induced from a character.)

The conditions of the following theorem are unfortunately rather complicated. We apologise for this. They deserve clarification. However the theorem does suffice for our purposes. The reason for introducing the submodules W_0 and W_1 of $\mathrm{ad}\,\overline{r}$ is that Ramakrishna's method [Ra2] may not work to kill cohomology classes on all of $\mathrm{ad}\,\overline{r}$. However sometimes in applications we will know for other reasons that there are no cohomology classes supported on these parts of $\mathrm{ad}\,\overline{r}$.

Theorem 2.6.3 *Keep the notation and assumptions of the start of section 2.4. In addition make the following assumptions.*

– *For all $v \in S$ the local deformation problem $\mathcal{D}_v$ is liftable and*

$$\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = \begin{cases} [F_v^+ : \mathbf{Q}_l]n(n-1)/2 & \text{if } v|l \\ 0 & \text{if } v\not|l. \end{cases}$$

– *For each infinite place v of F^+ we have $\chi(c_v) = -1$.*

– $\mathrm{ad}\,\overline{r}$ *and* $(\mathrm{ad}\,\overline{r})(1)$ *are semisimple $k[G_{F^+}]$-modules and have no irreducible constituent in common.*

– $H^i(\overline{r}(G_{F^+(\zeta_l)}), \mathfrak{g}_n(k)) = (0)$ *for $i = 0$ and 1.*

Suppose that W_0 and W_1 are G_{F^+}-submodules of $\mathrm{ad}\,\overline{r}$ with W_0 and $W_1(1)$ insubstantial. Suppose moreover that for all irreducible $k[G_{F^+,S}]$-submodules W and W' of $\mathfrak{g}_n(k)$ with $W' \not\subset W_0$ and $W \not\subset W_1$ we can find $\sigma \in G_{F,S}$ and $\alpha \in k^\times$ with the following properties:

– $\epsilon(\sigma) \not\equiv 1 \bmod l$.

– *The α generalised eigenspace $V_{\sigma,\alpha}$ of $\overline{r}(\sigma)$ and the $\alpha\epsilon(\sigma)$ generalised eigenspace $V_{\sigma,\alpha\epsilon(\sigma)}$ of $\overline{r}(\sigma)$ are one dimensional. Let $i_{\sigma,\alpha}$ (resp. $i_{\sigma,\alpha\epsilon(\sigma)}$) denote the inclusions $V_{\sigma,\alpha} \hookrightarrow k^n$ (resp. $V_{\sigma,\alpha\epsilon(\sigma)} \hookrightarrow k^n$). Let $\pi_{\sigma,\alpha} : k^n \to V_{\sigma,\alpha}$ (resp. $\pi_{\sigma,\alpha\epsilon(\sigma)} : k^n \to V_{\sigma,\alpha\epsilon(\sigma)}$) denote the σ-equivariant projections.*

– $i_{\sigma,\alpha\epsilon(\sigma)}\pi_{\sigma\alpha} \notin W_0$.

– $(i_{\sigma,\alpha\epsilon(\sigma)}\pi_{\sigma,\alpha\epsilon(\sigma)} - i_{\sigma,\alpha}\pi_{\sigma,\alpha}) \notin W_1$.

– $\pi_{\sigma,\alpha} \circ W \circ i_{\sigma,\alpha\epsilon(\sigma)} \neq (0)$.

– $\pi_{\sigma,\alpha} \circ w' \circ i_{\sigma,\alpha} \neq \pi_{\sigma,\alpha\epsilon(\sigma)} \circ w' \circ i_{\sigma,\alpha\epsilon(\sigma)}$ *for some $w' \in W'$.*

(We note that this property will also hold for any non-zero $\mathbf{F}_l[G_{F^+,S}]$-subspaces W and W' of $\mathfrak{g}_n(k)$ with $W' \not\subset W_0$ and $W \not\subset W_1$. Because it holds for W and W' if and only if it holds for their k-linear spans.)

Then we can find $(Q, \{\overline{\psi}_v\}_{v\in Q})$ as in definition 2.6.2 such that

$$R^{\mathrm{univ}}_{\mathcal{S}[Q]} = \mathcal{O}.$$

In particular there is a lifting $r : G_{F^+,S\cup Q} \to \mathcal{G}_n(\mathcal{O})$ of $\overline{r}$ unramified at all but finitely many primes, with $\nu \circ r = \chi$ and such that for all $v \in S$ the restriction $r|_{G_{F_{\widetilde{v}}}}$ lies in $\mathcal{D}_{\widetilde{v}}$.

Proof: We will continue to use the notation of definition 2.6.2. If the cohomology group $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\, \overline{r}(1)) = (0)$ then the proposition follows at once from corollary 2.3.6 (with $Q = \emptyset$). In the general case we need only show that we can find a prime $v \notin S$ of F^+ which splits in F such that

- $\mathbf{N}v \not\equiv 1 \bmod l$.
- $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{t}_v \oplus \overline{s}_v$ where $\overline{t}_v = \overline{\psi}_v \oplus \overline{\psi}_v \epsilon$ and neither $\overline{\psi}_v$ nor $\overline{\psi}_v \epsilon$ is a subquotient of $\overline{s}_v$.
-
$$\dim_k H^1_{(\mathcal{L}[\{v\}])^\perp}(G_{F^+,S\cup\{v\}}, (\mathrm{ad}\, \overline{r})(1)) < \dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(1)).$$
- $H^1_{\mathcal{S}[\{v\}]}(G_{F^+,S\cup\{v\}}, W_0) = (0)$ and $H^1_{(\mathcal{L}[\{v\}])^\perp}(G_{F^+,S\cup\{v\}}, W_1(1)) = (0)$, i.e. W_0 and $W_1(1)$ remain insubstantial for $\mathcal{S}[\{v\}]$.

(Then one can add primes v as above to S recursively until
$$H^1_{(\mathcal{L}[Q])^\perp}(G_{F^+,S\cup Q}, (\mathrm{ad}\, \overline{r})(1)) = (0).)$$

So let $v \notin S$ be a prime of F^+ which splits in F such that

- $\mathbf{N}v \not\equiv 1 \bmod l$.
- $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{t}_v \oplus \overline{s}_v$ where $\overline{t}_v = \overline{\psi}_v \oplus \overline{\psi}_v \epsilon$ and neither $\overline{\psi}_v$ nor $\overline{\psi}_v \epsilon$ is a subquotient of $\overline{s}_v$.
- $i_{\overline{\psi}_v \epsilon} \pi_{\overline{\psi}_v} \notin W_0$ and $i_{\overline{\psi}_v \epsilon} \pi_{\overline{\psi}_v \epsilon} - i_{\overline{\psi}_v} \pi_{\overline{\psi}_v} \notin W_1$.

Note that there are left exact sequences
$$(0) \to H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\, \overline{r}) \to H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\, \overline{r}) \to H^1(I_{F_{\widetilde{v}}}, k(i_{\overline{\psi}_v \epsilon} \pi_{\overline{\psi}_v}))$$
and
$$\begin{aligned}(0) \to H^1_{\mathcal{S}[\{v\}]}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\, \overline{r}) \to H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\, \overline{r}) \to \\ \to H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k(i_{\overline{\psi}_v \epsilon} \pi_{\overline{\psi}_v \epsilon} - i_{\overline{\psi}_v} \pi_{\overline{\psi}_v}))\end{aligned}$$
and
$$\begin{aligned}(0) \to H^1_{(\mathcal{L}[\{v\}]')^\perp}(G_{F^+,S\cup\{v\}}, (\mathrm{ad}\, \overline{r})(1)) \to H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(1)) \to \\ \to H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, ((\mathrm{ad}\, \overline{t})/k(i_{\overline{\psi}_v \epsilon} \pi_{\overline{\psi}_v}))(1)).\end{aligned}$$

It follows from lemma 2.3.4 (and the discussions of sections 2.4.7 and 2.4.8) that
$$\begin{aligned}&\dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\, \overline{r})(1)) - \dim_k H^1_{(\mathcal{L}[\{v\}]')^\perp}(G_{F^+,S\cup\{v\}}, (\mathrm{ad}\, \overline{r})(1)) \\ &= \dim_k H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\, \overline{r}) - \dim_k H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\, \overline{r}) + \\ &\quad + \dim_k L'_v - \dim H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}) \\ &= \dim_k H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\, \overline{r}) - \dim_k H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\, \overline{r}) + 1.\end{aligned}$$

Moreover because $i_{\overline{\psi}_v\epsilon}\pi_{\overline{\psi}_v} \notin W_0$ we see that $H^1(G_{F_{\tilde{v}}}, W_0) \cap L_v$ is contained in $H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, W_0)$ and so

$$H^1_{\mathcal{S}[\{v\}]}(G_{F^+,S\cup\{v\}}, W_0) \subset H^1_{\mathcal{S}}(G_{F^+,S}, W_0) = (0).$$

Similarly because $(i_{\overline{\psi}_v\epsilon}\pi_{\overline{\psi}_v\epsilon} - i_{\overline{\psi}_v}\pi_{\overline{\psi}_v}) \notin W_1$ we see that

$$H^1(G_{F_{\tilde{v}}}, W_1(1)) \cap L_v^\perp \subset H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, W_1(1))$$

and so

$$H^1_{\mathcal{L}[\{v\}]^\perp}(G_{F^+,S\cup\{v\}}, W_1(1)) \subset H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1(1)) = (0).$$

Thus the prime v will have the desired properties if

$$H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) \to H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, ((\mathrm{ad}\,\overline{t})/k(i_{\overline{\psi}_v\epsilon}\pi_{\overline{\psi}_v}))(1))$$

and

$$H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \hookrightarrow H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\,\overline{r}) \to H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, k(i_{\overline{\psi}_v\epsilon}\pi_{\overline{\psi}_v\epsilon} - i_{\overline{\psi}_v}\pi_{\overline{\psi}_v}))$$

are both non-trivial. (From the non-triviality of the first map we would then deduce that

$$\dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) \geq \dim_k H^1_{(\mathcal{L}[\{v\}]')^\perp}(G_{F^+,S\cup\{v\}}, (\mathrm{ad}\,\overline{r})(1)) + 1,$$

so that

$$\dim_k H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \geq \dim_k H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\,\overline{r})$$

and, in fact,

$$H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \xrightarrow{\sim} H^1_{\mathcal{S}[\{v\}]'}(G_{F^+,S\cup\{v\}}, \mathrm{ad}\,\overline{r}).)$$

Suppose that $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) \neq (0)$. It follows from lemma 2.3.4 that

$$\dim H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \dim H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) > 0.$$

Choose a non-zero class $[\varphi] \in H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1))$ and a non-zero class $[\varphi''] \in H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r})$. By the Cebotarev density theorem it suffices to show that we can choose $\sigma \in G_F$ and $\alpha \in k$ with the following properties.

- $\sigma|_{F(\zeta_l)} \neq 1$.
- $\overline{r}(\sigma)$ has eigenvalues α and $\alpha\epsilon(\sigma)$ and the corresponding generalised eigenspaces U and U' have dimension 1. Let i (resp. i') denote the inclusion of U (resp. U') into k^n and let π (resp. π') denote the σ-equivariant projection of k^n onto U (resp. U').
- $i'\pi \notin W_0$.
- $i'\pi' - i\pi \notin W_1$.
- $\pi \circ \varphi(\sigma) \circ i' \neq 0$.

– $\pi \circ \varphi''(\sigma) \circ i \neq \pi' \circ \varphi''(\sigma) \circ i'$.

Let L denote the extension of $F(\zeta_l)$ cut out by $\operatorname{ad} \overline{r}$. Replacing σ by $\sigma'\sigma$ with $\sigma' \in G_L$ we need only show that we can find $\sigma \in G_F$ and $\alpha \in k$ with the following properties.

– $\sigma|_{F(\zeta_l)} \neq 1$.
– $\overline{r}(\sigma)$ has eigenvalues α and $\alpha\epsilon(\sigma)$ and the corresponding generalised eigenspaces U and U' have dimension 1. Let i (resp. i') denote the inclusion of U (resp. U') into k^n and let π (resp. π') denote the σ-equivariant projection of k^n onto U (resp. U').
– $i'\pi \notin W_0$.
– $i'\pi' - i\pi \notin W_1$.
– $\pi \circ \varphi(G_L) \circ i' \neq 0$.
– $\sigma' \mapsto \pi \circ \varphi''(\sigma') \circ i - \pi' \circ \varphi''(\sigma') \circ i'$ is not identically zero on G_L.

Note that $\varphi(G_L) \not\subset W_0$ and $\varphi''(G_L) \not\subset W_1$ (because $H^1_{\mathcal{S}}(G_{F^+,S}, W_0) = (0)$ and $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1(1)) = (0)$). Hence the existence of σ follows from the assumptions of the theorem. $\Box$

Because the hypotheses of this theorem are so complicated we give a concrete illustration of the theorem. It will not be needed in the sequel. We will write $\mathrm{Cl}(F)$ for the class group of a number field F.

Corollary 2.6.4 *Suppose that $n > 1$ is an integer, that F^+ is a totally real field and that F is a totally imaginary quadratic extension of F^+. Suppose also that $l > n$ is a prime with the following properties.*

– *l is unramified in F^+.*
– *All primes of F^+ above l split in F.*
– *$l \not| \#\mathrm{Cl}(F)_{\mathrm{Gal}\,(F/F^+)}$, the order of the* $\mathrm{Gal}\,(F/F^+)$*-coinvariants* $\mathrm{Cl}(F)$.

Suppose finally that

$$\overline{r} : G_{F^+} \twoheadrightarrow \mathcal{G}_n(\mathbf{F}_l)$$

is a continuous, surjective homomorphism such that

– *$\overline{r}^{-1}(GL_n(\mathbf{F}_l) \times GL_1(\mathbf{F}_l)) = G_F$;*
– *$\overline{r}|_{G_F}$ only ramifies at primes which are split over F^+;*
– *$\nu \circ \overline{r}(c) = -1$ for any complex conjugation c;*
– *for any place w of F above l then $\overline{r}|_{G_{F_w}}$ is in the image of $\mathbf{G}_w$ and for each $i = 0, ..., l-2$ we have*

$$\dim_{k(w)} \mathrm{gr}^i \mathbf{G}_w^{-1} \overline{r}|_{G_{F_w}} \leq 1.$$

Then there is a finite extension $k/\mathbf{F}_l$ such that $\overline{r}$ lifts to a continuous homomorphism

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(W(k))$$

which ramifies at only finitely many primes and which is crystalline at all primes of F above l (with Hodge-Tate numbers all between 0 and $l-2$).

Proof: We apply the theorem. We take $\mathcal{O} = W(k)$ for a suitably large finite extension $k/\mathbf{F}_l$. We take S to be the set of places above l or below a prime of F at which $\overline{r}|_{G_F}$ is ramified. For $v|l$ we take $\mathcal{D}_v$ as in section 2.4.1. For $v \in S$ with $v \not| l$ we take $\mathcal{D}_v$ as in section 2.4.4. As $l > n$ we have $\mathrm{ad}\,\overline{r} = k1_n \oplus \mathrm{ad}^0\overline{r}$ and both summands are irreducible G_F-modules. As $F^+(\zeta_l)$ is linearly disjoint from F over F^+ (look at ramification above l) we have that $H^0(\overline{r}(G_{F^+(\zeta_l)}), k1_n) = (0)$ and $H^1(\overline{r}(G_{F^+(\zeta_l)}), k1_n) = (0)$. Clearly $H^0(\overline{r}(G_{F^+(\zeta_l)}), \mathfrak{g}_n^0(k)) = (0)$. By [CPS] (see table (4.5)) we have that $H^1(SL_n(\mathbf{F}_l), M_n(\mathbf{F}_l)^{\mathrm{tr}=0}) = (0)$, and so $H^1(\overline{r}(G_{F^+(\zeta_l)}), \mathfrak{g}_n^0(k)) = (0)$. We take $W_0 = k1_n$ and $W_1 = (k1_n)(1)$. Then

$$\begin{aligned} H^1_{\mathcal{S}}(G_{F^+,S}, W_0) &= \ker(H^1(G_{F^+}, k1_n) \longrightarrow \textstyle\bigoplus_v H^1(I_{F_{\widetilde{v}}}, k1_n)) \\ &= \ker(H^1(G_{F^+}, k1_n) \longrightarrow \textstyle\bigoplus_v H^1(I_{F^+_{\widetilde{v}}}, k1_n)) \\ &= \ker(H^1(G_F, k1_n) \longrightarrow \textstyle\bigoplus_{\widetilde{v}} H^1(I_{F_{\widetilde{v}}}, k1_n))^{\mathrm{Gal}\,(F/F^+)} \\ &= \mathrm{Hom}\,(\mathrm{Cl}(F)/(c-1)\mathrm{Cl}(F), k) = (0). \end{aligned}$$

(Note that if $\widetilde{v}$ is a prime of F ramified over F^+ then $H^1(I_{F^+_{\widetilde{v}}}, k1_n) \hookrightarrow H^1(I_{F_{\widetilde{v}}}, k1_n)$.) Also

$$H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1) = \ker(H^1(G_{F^+}, (k1_n)(1)) \longrightarrow \bigoplus_v H^1(I_{F^+_v}, (k1_n)(1))).$$

(Note that if $\widetilde{v}$ is a prime of F ramified over F^+ then $H^1(I_{F^+_{\widetilde{v}}}, (k1_n)(1)) \hookrightarrow H^1(I_{F_{\widetilde{v}}}, (k1_n)(1))$.) By, for instance, theorem 2.19 of [DDT] we see that

$$H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1) = (0).$$

The rest of the hypotheses of the theorem are easy to verify and the corollary follows. □

2.7. *An example.* — In this section we will specialise the theorem of the last section to the case where we will require it: $\overline{r}$ will be induced from a character and we will be looking for a lift r with the property that the restriction to some decomposition group corresponds (under the local Langlands correspondence) to a Steinberg representation.

Fix a positive *even* integer n and choose a second positive integer κ_n greater than $(n-1)((n+2)^{n/2}-(n-2)^{n/2})/2^{n+1}$. (This number is too large for its precise value to matter, what matters is that there is some constant κ_n which depends only on n which will suffice.)

In this section we will consider the following situation.

– $M/\mathbf{Q}$ is a Galois imaginary CM field of degree n with $\mathrm{Gal}\,(M/\mathbf{Q})$ cyclic generated by an element τ.
– $l > 1+4\kappa_n$ is a prime which splits completely in M and is $\equiv 1 \bmod n$.
– $Q \not\ni l$ is a finite set of rational primes, such that if $q \in Q$ then q splits completely in M and $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n$.
– $\overline{\theta} : \mathrm{Gal}\,(\overline{\mathbf{Q}}/M) \longrightarrow \overline{\mathbf{F}}_l^{\times}$ is a continuous character such that
  – $\theta\theta^c = \epsilon^{1-n}$;
  – there exists a prime $w|l$ of M such that for $i = 0, ..., n/2-1$ we have $\overline{\theta}|_{I_{\tau^i w}} = \epsilon^{-i}$;
  – if $v_1, ..., v_n$ are the primes of M above $q \in Q$ then $\{\overline{\theta}(\mathrm{Frob}_{v_i})\} = \{\alpha_q q^{-j} :\ j = 0, ..., n-1\}$ for some $\alpha_q \in \overline{\mathbf{F}}_l^{\times}$;
Let $S(\overline{\theta})$ denote the set of rational primes above which M or $\overline{\theta}$ is ramified. It includes l.
– $E/\mathbf{Q}$ is an imaginary quadratic field linearly disjoint from the normal closure of $\overline{M}^{\ker \overline{\theta}}(\zeta_l)/\mathbf{Q}$ in which every element of $S(\overline{\theta}) \cup Q$ splits; and such that the class number of E is not divisible by l.

The referee asks the good question: are there any examples where all these conditions are met? The answer is ‘yes’. One example is given in the proof of theorem 3.1 of [HSBT]. We remark that the primes in Q will be those at which the lift we construct will correspond (under the local Langlands correspondence) to a Steinberg representation.

Set $L/\mathbf{Q}$ equal to the normal closure over $\mathbf{Q}$ of the composite of E and $\overline{M}^{\ker \overline{\theta}}(\zeta_l)$. Also let $(EM)^+$ denote the maximal totally real subfield of EM. Then $\overline{\theta}|_{\mathrm{Gal}\,(L/EM)}$ extends to a homomorphism, which we will also denote $\overline{\theta}$,

$$\overline{\theta} : \mathrm{Gal}\,(L/(EM)^+) \longrightarrow \mathcal{G}_1(\overline{\mathbf{F}}_l)$$

such that $\overline{\theta}(c) = (1, 1, \jmath)$ and $\nu \circ \overline{\theta} = \epsilon^{1-n}$. Let $\overline{r} : \mathrm{Gal}\,(L/\mathbf{Q}) \to \mathcal{G}_n(\overline{\mathbf{F}}_l)$ denote the induction with multiplier ϵ^{1-n} from $(\mathrm{Gal}\,(L/(EM)^+), \mathrm{Gal}\,(L/EM))$ to $(\mathrm{Gal}\,(L/\mathbf{Q}), \mathrm{Gal}\,(L/E))$ of $\overline{\theta}$. (See section 2.1.)

We have an embedding

$$\begin{aligned} \mathrm{Gal}\,(L/EM) &\hookrightarrow (\overline{\mathbf{F}}_l^{\times})^{n/2} \times \mathbf{F}_l^{\times} \\ \alpha &\longmapsto (\overline{\theta}(\alpha), \overline{\theta}^{\tau}(\alpha), ..., \overline{\theta}^{\tau^{n/2-1}}(\alpha); \epsilon(\alpha)). \end{aligned}$$

Fix a primitive n^{th} root of unity $\zeta_n \in \mathbf{F}_l$. Suppose $\alpha = (\alpha_0, ..., \alpha_{n/2-1}) \in (\overline{\mathbf{F}}_l^\times)^{n/2}$ and $\beta \in \overline{\mathbf{F}}_l^\times$ satisfy $\beta^2 = \alpha_0 ... \alpha_{n/2-1}$. If $n/2 \leq i \leq n-1$ set $\alpha_i = \alpha_{i-n/2}^{-1}$. Let $\Gamma_{\alpha,\beta} = \Gamma$ denote the group generated by $(\overline{\mathbf{F}}_l^\times)^{n/2} \times \mathbf{F}_l^\times$ and two elements C and T satisfying

- $C^2 = 1$ and $T^n = 1$;
- $CTCT^{-1} = (\alpha_0, ..., \alpha_{n/2-1}; 1)$;
- $T(a_0, ..., a_{n/2-1}; b)T^{-1} = (a_1, ..., a_{n/2-1}, b^{1-n}a_0^{-1}; b)$;
- and $C(a_0, ..., a_{n/2-1}; b)C = (b^{1-n}a_0^{-1}, ..., b^{1-n}a_{n/2-1}^{-1}; b)$.

Define characters $\Xi : \Gamma \to \mathbf{F}_l^\times$ by

- $\Xi(T) = \zeta_n$,
- $\Xi(C) = -1$,
- and $\Xi(a_0, ..., a_{n/2-1}; b) = b$;

and $\Theta : \langle (\overline{\mathbf{F}}_l^\times \times \mathbf{F}_l^\times, CT^{n/2} \rangle \to \overline{\mathbf{F}}_l^\times$ such that

- $\Theta(a_0, ..., a_{n/2-1}; b) = a_0$,
- and $\Theta(CT^{n/2}) = \beta$.

Note that

- $\Theta(CT^iCT^{-i}) = \alpha_0 ... \alpha_{i-1}$ (because $(CTCT^{-1})T(CT^iCT^{-i})T^{-1} = CT^{i+1}CT^{-(i+1)}$), and
- $\Theta(T^iCT^{n/2}T^{-i}) = \beta(\alpha_0 ... \alpha_{i-1})^{-1}$ (because $(CT^iCT^{-i})T^i(CT^{n/2})T^{-i} = CT^{n/2}$).

Let $\Gamma_0 = \Gamma_{\alpha,\beta,0}$ denote the subgroup generated by $((\mathbf{F}_l^\times)^{\kappa_n})^{\oplus n/2+1}$ and by C and T. The next lemma tells us that for many calculations we can replace the group $\mathrm{Gal}\,(L/\mathbf{Q})$ by the more concrete groups Γ and Γ_0.

Lemma 2.7.1 *There exist α and β such that the embedding*

$$\mathrm{Gal}\,(L/EM) \hookrightarrow (\overline{\mathbf{F}}_l^\times)^{n/2} \times \mathbf{F}_l^\times$$

extends to an embedding

$$j : \mathrm{Gal}\,(L/\mathbf{Q}) \hookrightarrow \Gamma$$

satisfying

- *$\Xi \circ j = \epsilon$;*
- *$\Theta \circ j = \overline{\theta}$;*
- *the image of j contains Γ_0;*
- *some complex conjugation maps to C;*
- *and some lifting $\widetilde{\tau} \in \mathrm{Gal}\,(L/E)$ of the generator τ of $\mathrm{Gal}\,(EM/E) \xrightarrow{\sim} \mathrm{Gal}\,(M/\mathbf{Q})$ maps to T.*

If such an embedding exists for some α *it also exists for any element of* $\alpha((\mathbf{F}_l^\times)^{2\kappa_n})^{\oplus n/2}$.

Proof: Note that EM and $\mathbf{Q}(\zeta_l)$ are linearly disjoint over $\mathbf{Q}$. Thus we may choose a lifting $\widetilde{\tau} \in \mathrm{Gal}\,(L/E)$ of the generator τ of $\mathrm{Gal}\,(EM/E) \xrightarrow{\sim} \mathrm{Gal}\,(M/\mathbf{Q})$ with $\epsilon(\widetilde{\tau}) = \zeta_n$. Also choose a complex conjugation $c \in \mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$. Then $\epsilon(c\widetilde{\tau}^{n/2}) = 1$ and so

$$\begin{aligned}\overline{\theta}(\widetilde{\tau}^n) &= \overline{\theta}(c(c\widetilde{\tau}^{n/2})c(c\widetilde{\tau}^{n/2}))\\ &= (\overline{\theta}\overline{\theta}^c)(c\widetilde{\tau}^{n/2})\\ &= \epsilon(c\widetilde{\tau}^{n/2})^{1-n}\\ &= 1.\end{aligned}$$

Also note that $\epsilon(c\widetilde{\tau}c\widetilde{\tau}^{-1}) = 1$. Setting $\alpha_i = \overline{\theta}^{\tau^i}(c\widetilde{\tau}c\widetilde{\tau}^{-1})$ we get a homomorphism

$$j : \mathrm{Gal}\,(L/\mathbf{Q}) \hookrightarrow \Gamma$$

extending the embedding $\mathrm{Gal}\,(L/EM) \hookrightarrow (\overline{\mathbf{F}}_l^\times)^{n/2} \times \mathbf{F}_l^\times$ and which sends $\widetilde{\tau}$ to T and c to C. We have $\Xi \circ j = \epsilon$. Note that

$$\overline{\theta}(c\widetilde{\tau}^{n/2})^2 = \overline{\theta}(c\widetilde{\tau}^{n/2}c\widetilde{\tau}^{-n/2}) = \overline{\theta}(c\widetilde{\tau}c\widetilde{\tau}^{-1})\overline{\theta}^{\tau}(c\widetilde{\tau}c\widetilde{\tau}^{-1})...\overline{\theta}^{\tau^{n/2-1}}(c\widetilde{\tau}c\widetilde{\tau}^{-1}),$$

and so for some choice of β we have $\Theta \circ j = \overline{\theta}$.

Choose a place u of E above l. Let A denote the subgroup of the image of $\mathrm{Ind}_{\mathrm{Gal}\,(\overline{E}/EM)}^{\mathrm{Gal}\,(\overline{E}/E)}\overline{\theta}$ generated by the decomposition groups above u. Let w be a place of EM above u. For any integer i define β_i to be

- $-i_0$ if $i \equiv i_0 \bmod n$ and $0 \le i_0 \le n/2 - 1$, and
- $i_0 + 1 - 3n/2$ if $i \equiv i_0 \bmod n$ and $n/2 \le i_0 \le n-1$.

Note that $\beta_i + \beta_{i+n/2} = 1 - n$. We have

$$\prod_{i=0}^{n-1} I_{M_{\sigma^i w}} \twoheadrightarrow \prod_{i=0}^{n-1} \mathbf{F}_l^\times \twoheadrightarrow A \hookrightarrow (\overline{\mathbf{F}}_l^\times)^{n/2+1}.$$

The composite map

$$\prod_{i=0}^{n-1} \mathbf{F}_l^\times \longrightarrow (\overline{\mathbf{F}}_l^\times)^{n/2+1}$$

sends

$$(a_i)_i \longmapsto (\prod_{i=0}^{n-1} a_i^{\beta_i}, \prod_{i=0}^{n-1} a_i^{\beta_{i-1}}, ..., \prod_{i=0}^{n-1} a_i^{\beta_{i+1-n/2}}, (\prod_{i=0}^{n-1} a_i)^{1-n}).$$

Moreover by part three of lemma 2.7.2 below we see that the image has index dividing κ_n. Thus the image of j contains Γ_0.

Finally note that

$$((a_0, ..., a_{n/2-1}; 1)T)^n = 1$$

and

$$C(a_0, ..., a_{n/2-1}; 1)TC((a_0, ..., a_{n/2-1}; 1)T)^{-1} = (\alpha_0 a_0^{-2}, ..., \alpha_{n/2-1} a_{n/2-1}^{-2}; 1).$$

(These two equalities follow directly from the relations defining Γ:

$$\begin{aligned} &((a_0, ..., a_{n/2-1}; 1)T)^n \\ = &(a_0, ..., a_{n/2-1}; 1)(a_1, ..., a_{n/2-1}, a_0^{-1}; 1)...(a_0^{-1}, ..., a_{n/2-1}^{-1}; 1)... \\ &...(a_1^{-1}, ..., a_{n/2-1}^{-1}, a_0; 1)T^n \\ = &1; \end{aligned}$$

and

$$\begin{aligned} &C(a_0, ..., a_{n/2-1}; 1)C^{-1}(CTCT^{-1})(a_0, ..., a_{n/2-1}; 1)^{-1} \\ = &(a_0, ..., a_{n/2-1}; 1)^{-1}(\alpha_0, ..., \alpha_{n/2-1}; 1)(a_0, ..., a_{n/2-1}; 1)^{-1}.) \end{aligned}$$

$\square$

Here is the evaluation of a determinant that was used in the proof of the last lemma. The first two parts are only needed to help prove the third part.

Lemma 2.7.2 *We have the following evaluations of determinants.*

1. *For an $n \times n$ determinant:*

$$\det \begin{pmatrix} 1 \; b \; 0 \; 0 & & 0 \; 0 \\ 1 \; c \; b \; 0 & \dots & 0 \; 0 \\ 1 \; c \; c \; b & & 0 \; 0 \\ \vdots & \ddots & \vdots \\ 1 \; c \; c \; c & & c \; b \\ 1 \; c \; c \; c & \dots & c \; c \end{pmatrix} = (c-b)^{n-1}.$$

2. *For an $n \times n$ determinant:*

$$\det \begin{pmatrix} a \; b \; b \; b & & b \; b \\ c \; a \; b \; b & \dots & b \; b \\ c \; c \; a \; b & & b \; b \\ \vdots & \ddots & \vdots \\ c \; c \; c \; c & & a \; b \\ c \; c \; c \; c & \dots & c \; a \end{pmatrix} = (c(a-b)^n - b(a-c)^n)/(c-b).$$

3. For an $(n+1)\times(n+1)$ *determinant:*

$$\det\begin{pmatrix} 0 & 1 & 2 & 3 & & n-2 & n-1 & 2n-1 \\ n & 0 & 1 & 2 & \dots & n-3 & n-2 & 2n-1 \\ n+1 & n & 0 & 1 & & n-4 & n-3 & 2n-1 \\ & & \vdots & & \ddots & & \vdots & \\ 2n-3 & 2n-4 & 2n-5 & 2n-6 & & 0 & 1 & 2n-1 \\ 2n-2 & 2n-3 & 2n-4 & 2n-5 & \dots & n & 0 & 2n-1 \\ 2n-1 & 2n-1 & 2n-1 & 2n-1 & & 2n-1 & 2n-1 & 2(2n-1) \end{pmatrix}$$

$$= (-1)^n(2n-1)((n+1)^n + (n-1)^n)/2.$$

Proof: For the first part subtract the penultimate row from the last row, then the three from last row from the penultimate row and so on finally subtracting the first row from the second. One ends up with an upper triangular matrix.

For the second matrix let Δ_n denote the determinant. Subtract the first row from each of the others and expand down the last column. Using the first part, we obtain

$$\Delta_n = b(a-c)^{n-1} + (a-b)\det\begin{pmatrix} a & b & b & b & & b \\ c-a & a-b & 0 & 0 & \dots & 0 \\ c-a & c-b & a-b & 0 & & 0 \\ & & \vdots & & \ddots & \vdots \\ c-a & c-b & c-b & c-b & \dots & a-b \end{pmatrix}$$
$$= b(a-c)^{n-1} + (a-b)\Delta_{n-1}.$$

The second assertion follows easily by induction.

For the third matrix subtract the second row from the first, the third from the second and so on, finally subtracting the penultimate row from the two from last row. One obtains

$$\det\begin{pmatrix} -n & 1 & 1 & 1 & & 1 & 1 & 0 \\ -1 & -n & 1 & 1 & \dots & 1 & 1 & 0 \\ -1 & -1 & -n & 1 & & 1 & 1 & 0 \\ & & \vdots & & \ddots & & \vdots & \\ -1 & -1 & -1 & -1 & & -n & 1 & 0 \\ 2n-2 & 2n-3 & 2n-4 & 2n-5 & \dots & n & 0 & 2n-1 \\ 2n-1 & 2n-1 & 2n-1 & 2n-1 & & 2n-1 & 2n-1 & 2(2n-1) \end{pmatrix}.$$

Then add half the sum of the first $n-1$ rows to the penultimate row making it

$$n-1\ n-1\ n-1\ n-1\ \dots\ n-1\ (n-1)/2\ 2n-1.$$

Now subtract 1/2 of the last column from each of the first n columns. This leaves the first $n-1$ rows unchanged and the last two rows become

$$\begin{matrix} -1/2 & -1/2 & -1/2 & -1/2 & \dots & -1/2 & -n/2 & 2n-2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2(2n-1). \end{matrix}$$

Thus the determinant becomes

$$(2n-1)\det\begin{pmatrix} -n & 1 & 1 & 1 & & 1 & 1 \\ -1 & -n & 1 & 1 & \dots & 1 & 1 \\ -1 & -1 & -n & 1 & & 1 & 1 \\ & & \vdots & & \ddots & & \vdots \\ -1 & -1 & -1 & -1 & & -n & 1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -n \end{pmatrix}.$$

The result follows on applying the second part. □

There is a homomorphism

$$\widetilde{\Theta} : \langle (\overline{\mathbf{F}}_l^\times)^{n/2} \times \mathbf{F}_l^\times, C\rangle \longrightarrow \mathcal{G}_1(\overline{\mathbf{F}}_l^\times)$$

extending $\Theta|_{(\overline{\mathbf{F}}_l^\times)^{n/2}\times\mathbf{F}_l^\times}$ and with $\nu\circ\widetilde{\Theta} = \Xi^{1-n}$. It takes C to $(1,1,\jmath)$. Consider I, the induction of $\widetilde{\Theta}$ from $(\langle(\overline{\mathbf{F}}_l^\times)^{n/2}\times\mathbf{F}_l^\times, C\rangle, (\overline{\mathbf{F}}_l^\times)^{n/2}\times\mathbf{F}_l^\times)$ to $(\Gamma, \langle(\overline{\mathbf{F}}_l^\times)^{n/2}\times\mathbf{F}_l^\times, T\rangle)$ with multiplier Ξ^{1-n}. (See section 2.1.) Then I has a basis consisting of functions e_i for $i=0,...,n-1$ with $e_i(T^j)=\delta_{ij}$ for $j=0,...,n-1$. Let $f_0,...,f_{n-1}$ be the dual basis of $I^\vee$. If $(a_0,...,a_{n/2-1};b)\in(\overline{\mathbf{F}}_l^\times)^{n/2}\times\mathbf{F}_l^\times$ set $a_i = b^{1-n}a_{i-n/2}^{-1}$ for $i=n/2,...,n-1$. Then we have

- $Te_i = e_{i-1}$ (with $e_{-1}=e_{n-1}$);
- $(a_0,...,a_{n/2-1};b)e_i = a_ie_i$ for $i=0,...,n-1$;
- $Tf_i = f_{i-1}$;
- and $(a_0,...,a_{n/2-1};b)f_i = a_i^{-1}f_i$ for $i=0,...,n-1$.

Moreover

$$\langle e_i, e_j\rangle = \zeta_n^i\alpha_0...\alpha_{i-1}\delta_{ij}.$$

We have $\overline{r} = I\circ j$.

Then Γ acts on $\mathrm{ad}\, I$ via

- $Te_i\otimes f_j = e_{i-1}\otimes f_{j-1}$;
- $(a_0,...,a_{n/2-1};b)e_i\otimes f_j = a_i/a_je_i\otimes f_j$;
- $Ce_i\otimes f_j = -\zeta_n^{i-j}\alpha_j...\alpha_{i-1}e_j\otimes f_i$ if $0\le j\le i\le n-1$;
- and $Ce_i\otimes f_j = -\zeta_n^{i-j}(\alpha_i...\alpha_{j-1})^{-1}e_j\otimes f_i$ if $0\le i\le j\le n-1$.

Hence if $0\le i\le j\le n/2-1$ then

- $CT^{n/2}e_i \otimes f_j = -\zeta_n^{i-j}\alpha_i...\alpha_{j-1}e_{j+n/2} \otimes f_{i+n/2}$;
- $CT^{n/2}e_{j+n/2} \otimes f_{i+n/2} = -\zeta_n^{j-i}\alpha_i...\alpha_{j-1}e_i \otimes f_j$;
- $CT^{n/2}e_j \otimes f_i = -\zeta_n^{j-i}\alpha_i^{-1}...\alpha_{j-1}^{-1}e_{i+n/2} \otimes f_{j+n/2}$;
- $CT^{n/2}e_{i+n/2} \otimes f_{j+n/2} = -\zeta_n^{i-j}\alpha_i^{-1}...\alpha_{j-1}^{-1}e_j \otimes f_i$;
- $CT^{n/2}e_i \otimes f_{j+n/2} = \zeta_n^{i-j}\alpha_0^{-1}...\alpha_{i-1}^{-1}\alpha_j...\alpha_{n/2-1}e_j \otimes f_{i+n/2}$;
- $CT^{n/2}e_j \otimes f_{i+n/2} = \zeta_n^{j-i}\alpha_0^{-1}...\alpha_{i-1}^{-1}\alpha_j...\alpha_{n/2-1}e_i \otimes f_{j+n/2}$;
- $CT^{n/2}e_{i+n/2} \otimes f_j = \zeta_n^{i-j}\alpha_0...\alpha_{i-1}\alpha_j^{-1}...\alpha_{n/2-1}^{-1}e_{j+n/2} \otimes f_i$;
- and $CT^{n/2}e_{j+n/2} \otimes f_i = \zeta_n^{j-i}\alpha_0...\alpha_{i-1}\alpha_j^{-1}...\alpha_{n/2-1}^{-1}e_{i+n/2} \otimes f_j$.

For $j = 1, ..., n/2-1$ let $W_j^{\pm}$ denote the span of the vectors

$$e_i \otimes f_{i+j} \mp \zeta_n^{-j}e_{n/2+i+j} \otimes f_{n/2+i}$$

for $i = 0, ..., n-1$ (and where we consider the subscripts modulo n). Then $W_j^{\pm}$ is a Γ-invariant subspace of $\mathrm{ad}\, I$. The space W_j^+ is isomorphic to the induction from $\langle (\overline{\mathbf{F}}_l^{\times})^{n/2} \times \mathbf{F}_l^{\times}, CT^{n/2}\rangle$ to Γ of Θ/Θ^{T^j}. The space W_j^- is isomorphic to the induction from $\langle (\overline{\mathbf{F}}_l^{\times})^{n/2} \times \mathbf{F}_l^{\times}, CT^{n/2}\rangle$ to Γ of Θ/Θ^{T^j} times the order two character with kernel $(\overline{\mathbf{F}}_l^{\times})^{n/2} \times \mathbf{F}_l^{\times}$.

If χ is a character of $\Gamma/((\overline{\mathbf{F}}_l^{\times})^{n/2} \times \mathbf{F}_l^{\times})$ with $\chi(C) = -1$ let W_χ denote the span of

$$e_0 \otimes f_0 + \chi(T)e_1 \otimes f_1 + ... + \chi(T)^{n-1}e_{n-1} \otimes f_{n-1}.$$

Then W_χ is an Γ-invariant subspace of $\mathrm{ad}\, I$ on which Γ acts via χ.

Let $W_{n/2}$ denote the span of the vectors $e_i \otimes f_{i+n/2}$ for $i = 0, ..., n-1$ (with the subscripts taken modulo n). Then $W_{n/2}$ is a Γ-invariant subspace of $\mathrm{ad}\, I$ isomorphic to the induction from $\langle (\overline{\mathbf{F}}_l^{\times})^{n/2} \times \mathbf{F}_l^{\times}, CT^{n/2}\rangle$ to Γ of $\Theta/\Theta^{T^{n/2}}$. We have

$$\mathrm{ad}\, I = W_{n/2} \oplus (\bigoplus_\chi W_\chi) \oplus (\bigoplus_{j=1}^{n/2-1} W_j^+) \oplus (\bigoplus_{j=1}^{n/2-1} W_j^-).$$

Lemma 2.7.3 *The restrictions to $\Gamma_0^{\Xi=1}$ of the $2n-1$ representations $W_{n/2}$, $W_j^{\pm}$ (for $j = 1, ..., n/2-1$) and W_χ are all irreducible, non-trivial and pairwise non-isomorphic.*

Proof: It suffices to show the following:

- If $1 \leq j \leq n/2$ then $\Theta \neq \Theta^{T^j}$ on $((\overline{\mathbf{F}}_l^{\times})^{\kappa_n})^{\oplus n/2} \times \{1\}$.
- If $1 \leq j, j' \leq n/2$ and $0 \leq k \leq n-1$ then

$$\Theta/\Theta^{T^j} \neq \Theta^{T^k}/\Theta^{T^{j'+k}}$$

on $((\overline{\mathbf{F}}_l^{\times})^{\kappa_n})^{\oplus n/2} \times \{1\}$ unless $j = j'$ and $k = 0$.

These facts are easily checked because $(l-1)/\kappa_n > 4$. $\Box$

Proposition 2.7.4 *Keep the notation and assumptions listed at the start of this section. There is a continuous homomorphism*

$$r : G_{\mathbf{Q}} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbf{Q}}_l})$$

such that

- *r lifts $\overline{r}$;*
- *$\nu \circ r = \epsilon^{1-n}$;*
- *r is ramified at only finitely many primes, all of which split in E;*
- *for all places $v|l$ of E, $r|_{\mathrm{Gal}\,(\overline{E}_v/E_v)}$ is crystalline;*
- *for all $\tau \in \mathrm{Hom}\,(E, \overline{\mathbf{Q}}_l)$ above a prime $v|l$ of E;*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau, E_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{E}_v/E_v)} = 1$$

for $i = 0, ..., n-1$ and $= 0$ otherwise;

- *for any place v of E above a rational prime $q \in Q$, the restriction $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_v/E_v)}$ is unramified and $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_v/E_v)}(\mathrm{Frob}_v)$ has eigenvalues $\{\alpha q^{-j} : j = 0, ..., n-1\}$ for some $\alpha \in \overline{\mathbf{Q}}_l^{\times}$.*

Proof: Consider the following deformation problem for $\overline{r}$

$$\mathcal{S}_1 = (E/\mathbf{Q}, S_1, \widetilde{S}_1, \mathcal{O}, \overline{r}, \epsilon^{1-n}, \{\mathcal{D}_v\}_{v \in S_1}),$$

where $S_1 = Q \cup S(\overline{\theta})$ and $\mathcal{O}$ denotes the Witt vectors of $\overline{\mathbf{F}}_l$. For $v \in S_1$ we define $\mathcal{D}_v$ (and L_v) as follows.

- If $v = l$ the choice of $\mathcal{D}_v$ is described in section 2.4.1.
- If $v \in Q$ then $\mathcal{D}_v$ is as in section 2.4.5 with $m = n$ and $\widetilde{r}_v = 1$.
- If $v \in S(\overline{\theta}) - \{l\}$ then $\mathcal{D}_v$ is as in section 2.4.4.

Also set $W_0 = \bigoplus_\chi W_\chi \subset \mathrm{ad}\,\overline{r}$ and $\delta_{E/\mathbf{Q}} : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}\,(E/\mathbf{Q}) \cong \{\pm 1\}$.

Then $H^1_{\mathcal{S}_1}(G_{\mathbf{Q},S_1}, W_0)$ is the kernel of the map from $H^1(G_{\mathbf{Q}}, W_0)$ to

$$\bigoplus_{v \notin Q} H^1(I_{\mathbf{Q}_v}, W_0) \oplus \bigoplus_{v \in Q} \left(H^1(I_{\mathbf{Q}_v}, W_{\delta_{E/\mathbf{Q}}}) \oplus \bigoplus_{\chi \neq \delta_{E/\mathbf{Q}}} H^1(G_{\mathbf{Q}_v}, W_\chi) \right).$$

(To calculate the local condition at l use lemma 2.4.5 and corollary 2.4.4. To calculate the local condition at $v \in S(\overline{\theta}) - \{l\}$ use lemma 2.4.22. To calculate the local condition at $v \in Q$ use lemma 2.4.30.) Because l does not divide the order of the class group of E we see that

$$\ker \left(H^1(G_{\mathbf{Q}}, W_{\delta_{E/\mathbf{Q}}}) \longrightarrow \bigoplus_v H^1(I_{\mathbf{Q}_v}, W_{\delta_{E/\mathbf{Q}}}) \right) = (0).$$

On the other hand if $\chi \neq \delta_{E/\mathbf{Q}}$ then

$$\ker\left(H^1(G_{\mathbf{Q}}, W_\chi) \longrightarrow \bigoplus_{v \notin Q} H^1(I_{\mathbf{Q}_v}, W_\chi) \oplus \bigoplus_{v \in Q} H^1(G_{\mathbf{Q}_v}, W_\chi)\right)$$

is contained in $\mathrm{Hom}\,(\mathrm{Cl}_Q(EM), k)$, where $\mathrm{Cl}_Q(EM)$ denotes the quotient of the class group of EM by the subgroup generated by the classes of primes above elements of Q. Because the maximal elementary l extension of EM unramified everywhere is linearly disjoint from L over EM, the Cebotarev density theorem implies that we can enlarge Q so that $\mathrm{Hom}\,(\mathrm{Cl}_Q(EM), k) = (0)$. Make such an enlargement. Then $H^1_{\mathcal{L}_1}(G_{\mathbf{Q},S_1}, W_0) = (0)$.

Moreover $H^1_{\mathcal{L}_1^\perp}(G_{\mathbf{Q}}, W_{\delta_{E/\mathbf{Q}}}(1))$ is the kernel of the restriction map from $H^1(G_{\mathbf{Q}}, W_{\delta_{E/\mathbf{Q}}}(1))$ to

$$\left(H^1(G_{\mathbf{Q}_l}, W_{\delta_{E/\mathbf{Q}}}(1))/H^1(G_{\mathbf{Q}_l}/I_{\mathbf{Q}_l}, W_{\delta_{E/\mathbf{Q}}})^\perp\right) \oplus \bigoplus_{v \neq l} H^1(I_{\mathbf{Q}_v}, W_0).$$

From theorem 2.19 of [DDT] we deduce that

$$\# H^1_{\mathcal{L}_1^\perp}(G_{\mathbf{Q},S_1}, W_{\delta_{E/\mathbf{Q}}}(1)) = \# H^1_{\mathcal{L}_1}(G_{\mathbf{Q},S_1}, W_{\delta_{E/\mathbf{Q}}}) = 1,$$

i.e. $H^1_{\mathcal{L}_1^\perp}(G_{\mathbf{Q},S_1}, W_{\delta_{E/\mathbf{Q}}}(1)) = (0)$.

Now consider a second deformation problem

$$\mathcal{S}_2 = (E/\mathbf{Q}, S_2, \widetilde{S}_2, \mathcal{O}, \overline{r}, \epsilon^{1-n}, \{\mathcal{D}_v\}_{v \in S_2}).$$

Here $S_2 = S_1 \cup Q'$, where Q' will be a set of primes disjoint from S_1 such that if $q' \in Q'$ then

$$j(\mathrm{Frob}_{q'}) = T(a_0(q'), ..., a_{n/2-1}(q'); b(q'))$$

with $b(q')^n = 1$ and $\zeta_n b(q') \neq 1$. Thus the eigenvalues of $\overline{r}(\mathrm{Frob}_{q'})$ are the n^{th} roots of $b(q')^{n/2}$ each with multiplicity 1, and $\overline{\epsilon}(\mathrm{Frob}_{q'}) \neq 1$. Set $a_{i+n/2}(q') = b(q')^{1-n} a_i(q')^{-1}$ for $i = 0, ..., n/2-1$. For $v \in Q'$ choose an unramified character $\overline{\chi}_v$ of $G_{E_{\widetilde{v}}}$ with $\overline{\chi}_v(\mathrm{Frob}_{\widetilde{v}})^n = b(q')^{n/2}$, and let $\mathcal{D}_v$ and L_v be as in section 2.4.7 with $\overline{\chi} = \overline{\chi}_v$. Let π_v (resp. i_v, resp. π'_v, resp. i'_v) denote the projection onto the $\overline{\chi}_v(\mathrm{Frob}_{\widetilde{v}})$ (resp. inclusion of the $\overline{\chi}_v(\mathrm{Frob}_{\widetilde{v}})$, resp. projection onto the $b(q')\zeta_n\overline{\chi}_v(\mathrm{Frob}_v)$, resp. inclusion of the $b(q')\zeta_n\overline{\chi}_v(\mathrm{Frob}_{\widetilde{v}})$) eigenspace of $\mathrm{Frob}_{\widetilde{v}}$ in $\overline{r}$. Then $i'_v\pi_v$ is in the k-span of

$$\sum_{i,j=0}^{n-1} b(q')^i \zeta_n^i \overline{\chi}_v(\mathrm{Frob}_{\widetilde{v}})^{i-j} (a_1(q')...a_i(q'))^{-1} a_1(q')...a_j(q') e_i \otimes f_j.$$

Thus $i'_v\pi_v \notin W_0$ and so $H^1_{\mathcal{S}_2}(G_{\mathbf{Q},S_2}, W_0) \subset H^1_{\mathcal{S}_1}(G_{\mathbf{Q},S_1}, W_0) = (0)$.

On the other hand $i'_v\pi'_v - i_v\pi_v$ is in the k-linear span of

$$\sum_{i,j=0}^{n-1} ((b(q')\zeta_n)^{i-j} - 1)\overline{\chi}_v(\mathrm{Frob}_v)^{i-j}(a_1(q')...a_i(q'))^{-1}a_1(q')...a_j(q')e_i \otimes f_j$$

and so $i'_v\pi'_v - i_v\pi_v \notin W_0$ (because $b(q')\zeta_n \neq 1$). Thus

$$H^1_{\mathcal{L}_2^\perp}(G_{\mathbf{Q},S_2}, W_0(1)) = \ker\left(H^1_{\mathcal{L}_1^\perp}(G_{\mathbf{Q},S_1}, W_0(1)) \longrightarrow \bigoplus_{q'\in Q'} H^1(G_{\mathbf{Q}_{q'}}/I_{\mathbf{Q}_{q'}}, k)\right),$$

where the map onto the factor $H^1(G_{\mathbf{Q}_{q'}}/I_{\mathbf{Q}_{q'}}, k)$ is induced by $A \longmapsto \pi_v A i'_v$ for $v \in \widetilde{S}_2$ with $v|q'$, i.e. by

$$\sum_{i=0}^{n-1} x_i e_i \otimes f_i \longmapsto \sum_{i=0}^{n-1} x_i (b(q')\zeta_n)^i.$$

If $[\phi] \in H^1_{\mathcal{L}_1^\perp}(G_{\mathbf{Q},S_1}, W_0(1))$ then the extension P_ϕ of EM cut out by ϕ is nontrivial and l-power order and hence linearly disjoint from L over EM. Because $H^1_{\mathcal{L}_1^\perp}(G_{\mathbf{Q},S_1}, W_{\delta_{E/\mathbf{Q}}}(1)) = (0)$ we see that $\phi(\mathrm{Gal}\,(P_\phi/EM)) \not\subset W_{\delta_{E/\mathbf{Q}}}(1)$. Thus we can choose $b \neq \zeta_n^{-1}$ so that

$$\sum_{i=0}^{n-1} x_i e_i \otimes f_i \longmapsto \sum_{i=0}^{n-1} x_i (b\zeta_n)^i$$

is not identically zero on $\phi(\mathrm{Gal}\,(P_\phi/EM))$. Then choose $a_0, ..., a_{n/2-1} \in \overline{\mathbf{F}}_l^\times$ and $\sigma \in \mathrm{Gal}\,(LP_\phi/\mathbf{Q})$ such that $j(\sigma) = T(a_0, ..., a_{n/2-1}; b)$ and, if

$$\phi(\sigma) = \sum_{i=0}^{n-1} \phi_i(\sigma) e_i \otimes f_i$$

then

$$\sum_{i=0}^{n-1} (b\zeta_n)^i \phi_i(\sigma) \neq 0.$$

Let $q' \notin S_1$ be a rational prime unramified in LP_ϕ with $\mathrm{Frob}_{q'} = \sigma \in \mathrm{Gal}\,(LP_\phi/\mathbf{Q})$. Then if $q' \in Q'$ and $b(q') = b$ then $[\phi] \notin H^1_{\mathcal{L}_2^\perp}(G_{\mathbf{Q},S_2}, W_0(1))$. Thus we can choose Q' and the $b(q')$ for $q' \in Q'$ such that

$$H^1_{\mathcal{L}_2^\perp}(G_{\mathbf{Q},S_2}, W_0(1)) = (0).$$

Make such a choice.

Finally we will apply theorem 2.6.3 with $W_1 = W_0$ to complete the proof of the lemma. In the notation of theorem 2.6.3, given W and W', each equal to $W_{n/2}$ or some $W_j^{\pm}$, we will show that the conditions of theorem 2.6.3 can be verified with σ a lift of $T(a_0, ..., a_{n/2-1}; b) \in \Gamma_0$ for a suitable $a_0, ..., a_{n/2-1}, b$. We shall suppose that $b^n = 1$ but that $b \neq \zeta_n^{-1}$, so that $\epsilon(\sigma)^n = 1$ but $\epsilon(\sigma) \neq 1$. For $i = 0, ..., n/2-1$ write $a_{i+n/2} = b^{1-n} a_i^{-1}$. There is a decomposition

$$\overline{r} = \bigoplus_{\mu^n = b^{n/2}} V_\mu$$

into σ-eigenspaces, where σ acts on V_μ as μ and where V_μ is the span of

$$e_0 + \mu a_1^{-1} e_1 + ... + \mu^{n-1} a_1^{-1} ... a_{n-1}^{-1} e_{n-1}.$$

Let i_μ denote the inclusion $V_\mu \hookrightarrow \overline{r}$ and let π_μ denote the σ-equivariant projection $\overline{r} \twoheadrightarrow V_\mu$, so that $\pi_\mu i_\mu = \mathrm{Id}_{V_\mu}$. Note that

- $i_{\mu\epsilon(\sigma)} \pi_\mu = \sum_{i,j=0}^{n-1} a_1 ... a_j (a_1 ... a_i)^{-1} \mu^{i-j} \epsilon(\sigma)^i e_i \otimes f_j \notin W_0$
- and $i_{\mu\epsilon(\sigma)} \pi_{\mu\epsilon(\sigma)} - i_\mu \pi_\mu = \sum_{i,j=0}^{n-1} a_1 ... a_j (a_1 ... a_i)^{-1} \mu^{i-j} (\epsilon(\sigma)^{i-j} - 1) \notin W_0$.

Moreover

- $\pi_\mu (e_i \otimes f_{i+n/2}) i_{\mu\epsilon(\sigma)} = \epsilon(\sigma)^{i+n/2} \mu^{n/2} (a_{i+1} ... a_{i+n/2})^{-1}$;
- $\pi_\mu (e_i \otimes f_{i+j} \mp \zeta_n^{-j} e_{n/2+i+j} \otimes f_{n/2+i}) i_{\mu\epsilon(\sigma)} = (a_{i+1} ... a_{i+j})^{-1} \mu^j \epsilon(\sigma)^{i+j} (1 \pm b^{n/2} (\mu\zeta_n)^{-2j})$;
- $\pi_{\mu\epsilon(\sigma)} (e_i \otimes f_{i+n/2}) i_{\mu\epsilon(\sigma)} - \pi_\mu (e_i \otimes f_{i+n/2}) i_\mu = (\epsilon(\sigma)^{n/2} - 1) \mu^{n/2} (a_{i+1} ... a_{i+n/2})^{-1}$;
- and $\pi_{\mu\epsilon(\sigma)} (e_i \otimes f_{i+j} \mp \zeta_n^{-j} e_{n/2+i+j} \otimes f_{n/2+i}) i_{\mu\epsilon(\sigma)} - \pi_\mu (e_i \otimes f_{i+j} \mp \zeta_n^{-j} e_{n/2+i+j} \otimes f_{n/2+i}) i_\mu = (1 \pm (\zeta_n \mu)^{-2j}) (\epsilon(\sigma)^j - 1) \mu^j (a_{i+1} ... a_{i+j})^{-1}$.

Let β (resp. γ) denote a primitive $(n/2)^{th}$ (resp. $(2n)^{th}$) root of 1. Then we have:

- In the cases $W, W' \in \{W_{n/2}, W_1^-, ..., W_{n/2-1}^-\}$ taking $b = \mu = 1$ will satisfy the conditions of theorem 2.6.3.
- In the cases $W, W' \in \{W_{n/2}, W_1^+, ..., W_{n/2-1}^+\}$ taking $b = 1$ and $\mu = \zeta_n^{-1}$ will satisfy the conditions of theorem 2.6.3.
- If $W, W' \in \{W_1^{\pm}, ..., W_{n/2-1}^{\pm}\}$ taking $b = \zeta_n^{-1}\beta$ and $\mu = \zeta_n^{-1}\gamma$ will satisfy the conditions of theorem 2.6.3.

□

Lemma 2.7.5 *Keep the notation and the assumptions of the beginning of this section. Then* $\overline{r}(G_{F^+(\zeta_l)})$ *is big.*

Proof: This follows from lemmas 2.7.1 and 2.7.3, the fact that l does not divide $\#\bar{r}_0(G_{\mathbf{Q}})$ and the following calculations.

– Take $a_0 \in (\mathbf{F}_l^\times)^{\kappa_n}$ with $a_0^2 \neq 1$ and take $\sigma \in G_{F(\zeta_l)}$ with $j(\sigma) = (a_0, 1, ..., 1; 1) \in \Delta_0$. Then

$$\pi_{\sigma,a_0} W_\chi i_{\sigma,a_0} \neq (0).$$

– Take $(a_0, ..., a_{n/2-1}) \in (\mathbf{F}_l^\times)^{\oplus n/2}$ and $\sigma \in G_{F(\zeta_l)}$ with $j(\sigma) = T(a_0, ..., a_{n/2-1}; \zeta_n^{-1})$. Also take μ to be the product of ζ_n^{-1} with a primitive $(2n)^{th}$ root of 1. Set $a_{i+n/2} = \zeta_n^{-1} a_i$ for $i = 0, ..., n/2 - 1$. Then

$$\pi_{\sigma,\mu} e_i \otimes f_{i+n/2} i_{\sigma,\mu} = \mu^{n/2} (a_{i+1} ... a_{i+n/2})^{-1}$$

and

$$\pi_{\sigma,\mu}(e_i \otimes f_{i+j} \mp \zeta_n^{-j} e_{n/2+i+j} \otimes f_{n/2+i}) i_{\sigma,\mu} = (1 \mp (\mu\zeta_n)^{-2j}) \mu^j (a_{i+1} ... a_{i+j})^{-1}.$$

Thus $\pi_{\sigma,\mu} W_{n/2} i_{\sigma,\mu} \neq (0)$ and $\pi_{\sigma,\mu} W_j^\pm i_{\sigma,\mu} \neq (0)$.

□

3. Hecke algebras.

3.1. *GL_n over a local field: characteristic zero theory..* — In this section let p be a rational prime and let F_w be a finite extension of $\mathbf{Q}_p$. Let $\mathcal{O}_{F_w}$ denote the maximal order in F_w, let $\wp_w$ denote the maximal ideal in $\mathcal{O}_{F_w}$, let $k(w) = \mathcal{O}_{F_W}/\wp_w$ and let $q_w = \#k(w)$. We will use ϖ_w to denote a generator of $\wp_w$ in situations where the particular choice of generator does not matter. Fix a set $X = X(F_w)$ of representatives in $\mathcal{O}_{F_w}$ for $k(w)$ such that $0 \in X$. Also let $\overline{K}$ denote an algebraic closure of $\mathbf{Q}_l$. Also fix a positive integer n. We will write B_n for the Borel subgroup of GL_n consisting of upper triangular matrices.

We will use some, mostly standard, notation from [HT] without comment. For instance n-Ind, $\boxplus$, Sp_m, JL, rec and R_l. *On the other hand, if π is an irreducible smooth representation of $GL_n(F_w)$ over $\overline{K}$ we will use the notation $r_l(\pi)$ for the l-adic representation associated (as in [Tat]) to the Weil-Deligne representation*

$$\mathrm{rec}_l(\pi^\vee \otimes |\ |^{(1-n)/2}),$$

when it exists (i.e. when the eigenvalues of $\mathrm{rec}(\pi^\vee \otimes |\ |^{(1-n)/2})(\phi_w)$ are l-adic units for some lift ϕ_w of Frob_w). In [HT] we used $r_l(\pi)$ for the semisimplification of this representation.

For any integer $m \geq 0$ we will let $U_0(w^m)$ (resp. $U_1(w^m)$) denote the subgroup of $GL_n(\mathcal{O}_{F_w})$ consisting of matices whose last row is congruent to $(0, ..., 0, *)$ (resp. $(0, ..., 0, 1)$) modulo $\wp_w^m$. Thus $U_1(w^m)$ is a normal subgroup of $U_0(w^m)$ and we have a natural identification

$$U_0(w^m)/U_1(w^m) \cong (\mathcal{O}_{F_w}/\wp_w^m)^\times$$

by projection to the lower right entry of a matrix. We will also denote by $\mathrm{Iw}(w)$ the subgroup of $GL_n(\mathcal{O}_{F_w})$ consisting of matrices which are upper triangular modulo $\wp_w$ and by $\mathrm{Iw}_1(w)$ the subgroup of $\mathrm{Iw}(w)$ consisting of matrices whose diagonal entries are all congruent to one modulo $\wp_w$. Thus $\mathrm{Iw}_1(w)$ is a normal subgroup of $\mathrm{Iw}(w)$ and we have a natural identification

$$\mathrm{Iw}(w)/\mathrm{Iw}_1(w) \cong (k(w)^\times)^n,$$

under which $\mathrm{diag}(\alpha_1, ..., \alpha_n)$ maps to $(\alpha_1 \bmod \wp_w, ..., \alpha_n \bmod \wp_w)$.

We will let $\varsigma_{w,j}$ denote the matrix

$$\begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix}.$$

We will also let w_m denote the $m \times m$-matrix with $(w_m)_{ij} = 1$ if $i+j = m+1$ and $(w_m)_{ij} = 0$ otherwise. Finally we will let $w_{n,i}$ denote the matrix

$$\begin{pmatrix} 1_{i-1} & 0 \\ 0 & w_{n+1-i} \end{pmatrix}.$$

For $j = 1, ..., n$ let $T_w^{(j)}$ denote the Hecke operator

$$[GL_n(\mathcal{O}_{F_w})\varsigma_{w,j}GL_n(\mathcal{O}_{F_w})].$$

For $j = 1, ..., n-1$ and for $m > 0$ let $U_w^{(j)}$ denote the Hecke operator

$$[U_0(w^m)\varsigma_{w,j}U_0(w^m)]$$

or

$$[U_1(w^m)\varsigma_{w,j}U_1(w^m)].$$

If W is a smooth representation of $GL_n(F_w)$ and if $m_1 > m_2 > 0$ then the action of $U_w^{(j)}$ is compatible with the inclusions

$$W^{U_0(w^{m_2})} \subset W^{U_1(w^{m_2})} \subset W^{U_1(w^{m_1})}.$$

(This follows easily from the coset decompositions

$$U_1(w^m)\varsigma_{w,j}U_1(w^m) = \coprod_{I,b} bU_1(w^m)$$

and

$$U_0(w^m)\varsigma_{w,j}U_0(w^m) = \coprod_{I,b} bU_0(w^m)$$

where I runs over j element subsets of $\{1, ..., n-1\}$ and b runs over elements of $B_n(F_w)$ with

- $b_{rr} = \varpi_w$ if $r \in I$ and $= 1$ otherwise,
- $b_{rs} \in X$ if $s > r$, and $= 0$ unless $r \in I$ and $s \notin I$.

See [Man1].)

If $\alpha \in F_w^\times$ has non-negative valuation we will write V_α for the Hecke operators

$$[U_0(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_0(w)]$$

and

$$[U_1(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_1(w)].$$

If W is a smooth representation of $GL_n(F_w)$ then the action of V_α is compatible with the inclusion

$$W^{U_0(w)} \subset W^{U_1(w)}.$$

(This follows from the easily verified equalities

$$U_1(w)\left(U_0(w) \cap \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix} U_0(w) \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) = U_0(w)$$

and

$$\begin{aligned} & U_1(w) \cap \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix} U_0(w) \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \\ = & U_1(w) \cap \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix} U_1(w) \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.) \end{aligned}$$

It is well known that there is an isomorphism

$$\mathbf{Z}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \cong \mathbf{Z}[T_1, T_2, ..., T_n, T_n^{-1}],$$

under which T_j corresponds to $T_w^{(j)}$. (The latter ring is the polynomial algebra in the given variables.) Alternatively we have the Satake isomorphism

$$\mathbf{Z}[1/q_w][GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \cong \mathbf{Z}[1/q_w][X_1^{\pm 1}, ..., X_n^{\pm 1}]^{S_n},$$

under which $T_w^{(j)}$ corresponds to $q_w^{j(1-j)/2} s_j(X_1, ..., X_n)$, where s_j is the j^{th} elementary symmetric function (i.e. the sum of all square free monomials of degree j). This is *not* the standard normalisation of the Satake isomorphism.

The next lemma is well known. We include a proof partly to establish notation and partly as a warm up for later calculations of a similar nature.

Lemma 3.1.1 *Suppose that $\chi_1, ..., \chi_n$ are unramified characters of $F_w^\times$. Then $(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{GL_n(\mathcal{O}_{F_w})}$ is one dimensional and $T_w^{(j)}$ acts on it by $q_w^{j(n-j)/2} s_j(\chi_1(\varpi_w), ..., \chi_n(\varpi_w))$, where s_j is the j^{th} elementary symmetric function (i.e. the sum of all square free monomials of degree j). If*

$$T \in \mathbf{Z}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

has Satake transform $P(X_1, ..., X_n)$ then the eigenvalue of T on

$$(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{GL_n(\mathcal{O}_{F_w})}$$

is $P(q_w^{(n-1)/2}\chi_1(\varpi_w), ..., q_w^{(n-1)/2}\chi_n(\varpi_w))$.

Proof: The fixed space $(\text{n-Ind}\,^{GL_n(F_w)}_{B_n(F_w)}(\chi_1,...,\chi_n))^{GL_n(\mathcal{O}_{F_w})}$ is spanned by the function φ_0 where

$$\varphi_0(bu)=\prod_{i=1}^n \chi_i(b_{ii})|b_{ii}|^{(n+1)/2-i}$$

for $b\in B_n(F_w)$ and $u\in GL_n(\mathcal{O}_{F_w})$. Then $(T_w^{(j)}\varphi_0)(1)$ equals the eigenvalue of $T_w^{(j)}$ on $(\text{n-Ind}\,^{GL_n(F_w)}_{B_n(F_w)}(\chi_1,...,\chi_n))^{GL_n(\mathcal{O}_{F_w})}$. But

$$(T_w^{(j)}\varphi_0)(1)=\sum_I\sum_b \varphi_0(b)$$

where I runs over j element subsets of $\{1,...,n\}$ and b runs over elements of $B_n(F_w)$ with

- $b_{rr}=\varpi_w$ if $r\in I$ and $b_{rr}=1$ otherwise;
- if $s>r$, $r\in I$ and $s\notin I$ then $b_{rs}\in X$;
- if $s>r$ and either $r\notin I$ or $s\in I$ then $b_{rs}=0$.

Thus

$$\begin{aligned}(T_w^{(j)}\varphi_0)(1)&=\textstyle\sum_I q_w^{\sum_{k=1}^j(n-j+k-i_k)}\prod_{i\in I}\chi_i(\varpi_w)q_w^{i-(n+1)/2}\\ &=q_w^{j(n-j)/2}\textstyle\sum_I\prod_{i\in I}\chi_i(\varpi_w),\end{aligned}$$

where $I=\{i_1<...<i_j\}$ runs over j element subsets of $\{1,...,n\}$. The lemma follows. $\Box$

Corollary 3.1.2 *Suppose that π is an unramified irreducible admissible representation of $GL_n(F_w)$ over $\overline{K}$. Let $t_\pi^{(j)}$ denote the eigenvalue of $T_w^{(j)}$ on $\pi^{GL_n(\mathcal{O}_{F_w})}$. Then $r_l(\pi)^\vee(1-n)(\mathrm{Frob}_w)$ has characteristic polynomial*

$$X^n-t_\pi^{(1)}X^{n-1}+...+(-1)^jq_w^{j(j-1)/2}t_\pi^{(j)}X^{n-j}+...+(-1)^nq_w^{n(n-1)/2}t_\pi^{(n)}.$$

Proof: Suppose that $\pi=\chi_1\boxplus...\boxplus\chi_n$. Then

$$r_l(\pi)^\vee(1-n)=\bigoplus_i(\chi_i|\ \ |^{(1-n)/2})\circ \mathrm{Art}^{-1},$$

so that $r_l(\pi)^\vee(1-n)(\mathrm{Frob}_w)$ has characteristic polynomial

$$(X-\chi_1(\varpi_w)q_w^{(n-1)/2})...(X-\chi_n(\varpi_w)q_w^{(n-1)/2}).$$

$\Box$

Lemma 3.1.3 *Suppose that π is an unramified irreducible admissible representation of $GL_n(F_w)$ over $\overline{K}$. Let $t_\pi^{(j)}$ denote the eigenvalue of $T_w^{(j)}$ on $\pi^{GL_n(\mathcal{O}_{F_w})}$. Then $\pi^{U_0(w)} = \pi^{U_1(w)}$ and the characteristic polynomial of V_{ϖ_w} on $\pi^{U_0(w)}$ divides*

$$X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}.$$

Proof: The first assertion is immediate because the central character of π is unramified. Choose unramified characters $\chi_i : F_w^\times \to \overline{K}^\times$ for $i = 1, .., n$ such that the $q_w^{(n-1)/2} \chi_i(\varpi_w)$ are the roots of

$$X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}$$

with multiplicities. From the last lemma we see that π is a subquotient of n-Ind$_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)$. Thus it suffices to show that the eigenvalues of V_ϖ on n-Ind$_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)^{U_0(w)}$ are $\{q_w^{(n-1)/2} \chi_i(\varpi_w)\}$, with multiplicities (as roots of the characteristic polynomial).

The space n-Ind$_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)^{U_0(w)}$ has a basis of functions φ_i for $i = 1, ..., n$ where the support of φ_i is contained in $B_n(F_w) w_{n,i} U_0(w)$ and $\varphi_i(w_{n,i}) = 1$. We have

$$V_{\varpi_w} \varphi_i = \sum_j (V_{\varpi_w} \varphi_i)(w_{n,j}) \varphi_j.$$

But

$$\begin{aligned}
(V_{\varpi_w}\varphi_i)(w_{n,j}) &= \textstyle\sum_{x \in X^{n-1}} \varphi_i \left(w_{n,j} \begin{pmatrix} 1_{n-1} & 0 \\ \varpi_w x & \varpi_w \end{pmatrix} \right) \\
&= \textstyle\sum_{x \in X^{j-1}} \sum_{y \in X^{n-j}} \varphi_i \begin{pmatrix} 1_{j-1} & 0 & 0 \\ \varpi_w x & \varpi_w y & \varpi_w \\ 0 & w_{n-j} & 0 \end{pmatrix} \\
&= q_w^{n-j} q_w^{j-(n+1)/2} \chi_j(\varpi_w) \textstyle\sum_{x \in X^{j-1}} \varphi_i \begin{pmatrix} 1_{j-1} & 0 & 0 \\ x & 0 & 1 \\ 0 & w_{n-j} & 0 \end{pmatrix}
\end{aligned}$$

A matrix $g \in GL_n(\mathcal{O}_{F_w})$ lies in $B_n(\mathcal{O}_{F_w}) w_{n,i} U_0(w)$ if and only if i is the largest integer such that $(0, ..., 0, 1)$ lies in the $k(w)$ span of the reduction modulo $\wp_w$ of the last $n+1-i$ rows of g. Thus

$$(V_{\varpi_w}\varphi_i)(w_{n,j})$$

is

– 0 if $i > j$,

– $q_w^{(n-1)/2}\chi_j(\varpi_w)$ if $i=j$, and
– $(q_w-1)q_w^{j-i-1}q_w^{(n-1)/2}\chi_j(\varpi_w)$ if $i<j$.

Thus the matrix of V_{ϖ_w} with respect to the basis $\{\varphi_i\}$ of the space $\text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1,\dots,\chi_n)^{U_0(w)}$ is triangular with diagonal entries $q_w^{(n-1)/2}\chi_j(\varpi_w)$. The lemma follows. □

Lemma 3.1.4 *Suppose that we have a partition $n=n_1+n_2$ and that π_1 (resp. π_2) is a smooth representation of $GL_{n_1}(F_w)$ (resp. $GL_{n_2}(F_w)$). Let $P\supset B_n$ denote the parabolic corresponding to the partition $n=n_1+n_2$. Set $\pi=\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\pi_1\otimes\pi_2)$. Then*

$$\pi^{U_1(w)}\cong\left(\pi_1^{GL_{n_1}(\mathcal{O}_{F_w})}\otimes\pi_2^{U_1(w)}\right)\oplus\left(\pi_1^{U_1(w)}\otimes\pi_2^{GL_{n_2}(\mathcal{O}_{F_w})}\right).$$

Moreover $U_w^{(j)}$ acts as

$$\begin{pmatrix} A & 0\\ * & B\end{pmatrix}$$

where

$$A=\sum_{j_1+j_2=j}q_w^{(n_1j_2+n_2j_1)/2-j_1j_2}(T_w^{(j_1)}\otimes U_w^{(j_2)})$$

and

$$B=\sum_{j=j_1+j_2}q_w^{(n_1j_2+n_2j_1)/2-j_1j_2}(U_w^{(j_1)}\otimes T_w^{(j_2)})$$

and if $\alpha\in F_w^\times$ has positive valuation then V_α acts as

$$\begin{pmatrix} |\alpha|^{-n_1/2}(1\otimes V_\alpha) & *\\ 0 & |\alpha|^{-n_2/2}(V_\alpha\otimes 1)\end{pmatrix}.$$

Proof: Let

$$\omega=\begin{pmatrix} 1_{n_1-1} & 0 & 0\\ 0 & 0 & 1\\ 0 & 1_{n_2} & 0\end{pmatrix}.$$

Then, by the Bruhat decomposition,

$$GL_n(F_w)=P(F_w)U_1(w)\coprod P(F_w)\omega U_1(w)$$

so that

$$\begin{aligned}&(\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}\pi_1\otimes\pi_2)^{U_1(w)}\\ &=(\pi_1\otimes\pi_2)^{P(F_w)\cap U_1(w)}\oplus(\pi_1\otimes\pi_2)^{P(F_w)\cap\omega U_1(w)\omega^{-1}}\\ &=\pi_1^{GL_{n_1}(\mathcal{O}_{F_w})}\otimes\pi_2^{U_1(w)}\oplus\pi_1^{U_1(w)}\otimes\pi_2^{GL_{n_2}(\mathcal{O}_{F_w})}.\end{aligned}$$

Specifically $x \in \pi_1^{GL_{n_1}(\mathcal{O}_{F_w})} \otimes \pi_2^{U_1(w)}$ corresponds to a function φ_x supported on $P(F_w)U_1(w)$ with $\varphi_x(1) = x$, and $y \in \pi_1^{U_1(w)} \otimes \pi_2^{GL_{n_2}(\mathcal{O}_{F_w})}$ corresponds to a function φ'_y supported on $P(F_w)\omega U_1(w)$ with $\varphi'_y(\omega) = y$.

If $\varphi \in (\text{n-Ind}_{P(F_w)}^{GL_n(F_w)} \pi_1 \otimes \pi_2)^{U_1(w)}$ then

$$(U_w^{(j)}\varphi)(a) = \sum_I \sum_b \varphi(ab)$$

where I runs over j element subsets of $\{1, ..., n-1\}$ and where b runs over elements of $B_n(F_w)$ with

- $b_{rr} = \varpi_w$ if $r \in I$ and $= 1$ otherwise,
- $b_{rs} \in X$ if $s > r$, and $= 0$ unless $r \in I$ and $s \notin I$.

Thus

$$(U_w^{(j)}\varphi'_y)(1) = \sum_I \sum_b \varphi'_y(b) = 0$$

and

$$(U_w^{(j)}\varphi_x)(1) = \sum_{I_1,I_2} \sum_{a,b,c} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x$$

where I_1 runs over subsets of $\{1, ..., n_1\}$, I_2 runs over subsets of $\{1, ..., n_2-1\}$, $a \in B_{n_1}(F_w)$, $b \in M_{n_1 \times n_2}(F_w)$ and $c \in B_{n_2}(F_w)$ such that

- $\#I_1 + \#I_2 = j$,
- $a_{rr} = \varpi_w$ if $r \in I_1$ and $= 1$ otherwise,
- $c_{rr} = \varpi_w$ if $r \in I_2$ and $= 1$ otherwise,
- if $s > r$ then $a_{rs} \in X$ and $= 0$ unless $r \in I_1$ and $s \notin I_1$,
- if $s > r$ then $c_{rs} \in X$ and $= 0$ unless $r \in I_2$ and $s \notin I_2$,
- $b_{rs} \in X$ and $= 0$ unless $r \in I_1$ and $s \notin I_2$.

Equivalently

$$(U_w^{(j)}\varphi_x)(1) = \sum_{j_1+j_2=j} q_w^{(n_1 j_2 + n_2 j_1)/2 - j_1 j_2} (T_w^{(j_1)} \otimes U_w^{(j_2)})x.$$

Similarly

$$(U_w^{(j)}\varphi'_y)(\omega) = \sum_{I_1,I_2} \sum_{a,b,c,d,e} \varphi'_y(\begin{pmatrix} a & c & b \\ 0 & 1 & 0 \\ 0 & e & d \end{pmatrix} \omega),$$

where $I_1 \subset \{1, ..., n_1 - 1\}$, $I_2 \subset \{1, ..., n_2\}$, $a \in B_{n_1-1}(F_w)$, $b \in M_{(n_1-1)\times n_2}(F_w)$, $c \in F_w^{n_1-1}$, $d \in B_{n_2}(F_w)$ and $e \in F_w^{n_2}$ with

- $\#I_1 + \#I_2 = j$,

- $a_{rr} = \varpi_w$ if $r \in I_1$ and $= 1$ otherwise,
- $d_{rr} = \varpi_w$ if $r \in I_2$ and $= 1$ otherwise,
- if $s > r$ then $a_{rs} \in X$ and $= 0$ unless $r \in I_1$ and $s \notin I_1$,
- if $s > r$ then $d_{rs} \in X$ and $= 0$ unless $r \in I_2$ and $s \notin I_2$,
- $b_{rs} \in X$ and $= 0$ unless $r \in I_1$ and $s \notin I_2$,
- $c_r \in X$ and $= 0$ unless $r \in I_1$,
- $e_r \in X$ and $= 0$ unless $r \in I_2$.

The matrix

$$\begin{pmatrix} a & c & b \\ 0 & 1 & 0 \\ 0 & e & d \end{pmatrix} \omega \in P(F_w)\omega U_1(w)$$

if and only if

$$\begin{pmatrix} a & c & b \\ 0 & 1 & 0 \\ 0 & d^{-1}e & 1_{n_2} \end{pmatrix} \in P(F_w)\omega U_1(w)\omega^{-1}$$

if and only if $e = 0$. Thus

$$(U_w^{(j)}\varphi'_y)(\omega) = \sum_{j=j_1+j_2} q_w^{(n_1 j_2 + n_2 j_1)/2 - j_1 j_2} (U_w^{(j_1)} \otimes T_w^{(j_2)}) y.$$

Now suppose $\alpha \in F_w^\times$ has non-negative valuation. If φ is an element of $(\text{n-Ind}_{P(F_w)}^{GL_n(F_w)} \pi_1 \otimes \pi_2)^{U_1(w)}$ then

$$(V_\alpha \varphi)(a) = \sum_{b \in (\mathcal{O}_{F_w}/(\alpha))^{n-1}} \varphi(a \begin{pmatrix} 1_{n-1} & 0 \\ \varpi_w b & \alpha \end{pmatrix}).$$

Thus

$$(V_\alpha \varphi_x)(1) = \sum_{b \in (\mathcal{O}_{F_w}/(\alpha))^{n_1}} \sum_{c \in (\mathcal{O}_{F_w}/(\alpha))^{n_2 - 1}} \varphi_x \begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ \varpi_w b & \varpi_w c & \alpha \end{pmatrix}.$$

However

$$\begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ \varpi_w b & \varpi_w c & \alpha \end{pmatrix} \in P(F_w)U_1(w)$$

if and only if

$$\begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ \alpha^{-1}\varpi_w b & 0 & 1 \end{pmatrix} \in P(F_w)U_1(w)$$

if and only if $b=0$. Hence

$$\begin{aligned}(V_\alpha\varphi_x)(1) &= \textstyle\sum_{c\in(\mathcal{O}_{F_w}/(\alpha))^{n_2-1}} \varphi_x \begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ 0 & \varpi_w c & \alpha \end{pmatrix} \\ &= |\alpha|^{-n_1/2}(1\otimes V_\alpha)x.\end{aligned}$$

On the other hand

$$(V_\alpha\varphi_x)(\omega) = \sum_{b\in(\mathcal{O}_{F_w}/(\alpha))^{n_1-1}} \sum_{c\in(\mathcal{O}_{F_w}/(\alpha))^{n_2}} \varphi_x \left(\begin{pmatrix} 1_{n_1-1} & 0 & 0 \\ \varpi_w b & \alpha & \varpi_w c \\ 0 & 0 & 1_{n_2} \end{pmatrix} \omega \right) = 0.$$

Similarly

$$\begin{aligned}(V_\alpha\varphi'_y)(\omega) &= \textstyle\sum_{b\in(\mathcal{O}_{F_w}/(\alpha))^{n_1-1}} \sum_{c\in(\mathcal{O}_{F_w}/(\alpha))^{n_2}} \varphi'_y \left(\begin{pmatrix} 1_{n_1-1} & 0 & 0 \\ \varpi_w b & \alpha & \varpi_w c \\ 0 & 0 & 1_{n_2} \end{pmatrix} \omega \right) \\ &= |\alpha|^{-n_2/2}(V_\alpha\otimes 1)y.\end{aligned}$$

The lemma follows. □

Lemma 3.1.5 *Suppose that π is an irreducible admissible representation of $GL_n(F_w)$ over $\overline{K}$ with a $U_1(w)$ fixed vector but no $GL_n(\mathcal{O}_{F_w})$-fixed vector. Then $\dim \pi^{U_1(w)}=1$ and there is a character with open kernel, $V_\pi : F_w^\times \to \overline{K}^\times$ such that $V_\pi(\alpha)$ is the eigenvalue of V_α on $\pi^{U_1(w)}$ for all $\alpha\in F_w^\times$ with non-negative valuation. For $j=1,...,n-1$, let $u_\pi^{(j)}$ denote the eigenvalue of $U_w^{(j)}$ on $\pi^{U_1(w)}$ and define $Q_\pi^{\mathrm{nr}}(X)\in\overline{K}[X]$ to be*

$$X^{n-1}-u_\pi^{(1)}X^{n-2}+...+(-1)^j q_w^{j(j-1)/2}u_\pi^{(j)}X^{n-1-j}+...+(-1)^n q_w^{(n-1)(n-2)/2}u_\pi^{(n-1)}.$$

Then there is an exact sequence

$$(0)\to s\to r_l(\pi)^\vee(1-n)\to V_\pi\circ \mathrm{Art}_{F_w}^{-1}\to(0)$$

where s is unramified and $s(\mathrm{Frob}_w)$ has characteristic polynomial $P_\pi^{\mathrm{nr}}(X)$. If $\pi^{U_0(w)}\neq(0)$ then $q_w^{-1}V_\pi(\varpi_w)$ is a root of $P_\pi^{\mathrm{nr}}(X)$. If, on the other hand, $\pi^{U_0(w)}=(0)$ then $r_l(\pi)^\vee(1-n)(\mathrm{Gal}\,(\overline{F}_w/F_w))$ is abelian.

Proof: If π is an irreducible, cuspidal, smooth representation of $GL_m(F_w)$ then the conductor of $\mathrm{rec}(\pi)\geq m$ unless $m=1$ and π is unramified. If π is an irreducible, square integrable, smooth representation of $GL_m(F_w)$ then the conductor of $\mathrm{rec}(\pi)\geq m$ unless $\pi=\mathrm{Sp}_m(\chi)$ for some unramified character χ, in which case the conductor is $m-1$. As any irreducible, square integrable, smooth representation π of $GL_m(F_w)$ is generic we see from [JPSS] that $\pi^{U_1(w)} \neq (0)$ if and only if either $m=1$ and π has conductor ≤ 1, or $m=2$ and $\pi = \mathrm{Sp}_2(\chi)$ for some unramified character χ of $F_w^\times$.

Now suppose that $n = n_1 + ... + n_r$ is a partition of n and let $P \supset B_n$ denote the corresponding parabolic. Let π_i be an irreducible, square integrable, smooth representation of $GL_{n_i}(F_w)$. If

$$(\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi_1 \otimes ... \otimes \pi_r)^{U_1(w)} \neq (0)$$

then by the last lemma there must exist an index i_0 such that:

– For $i \neq i_0$ we have $n_i = 1$ and π_i unramified.

– Either $n_{i_0} = 1$ and π_{i_0} has conductor ≤ 1 or $n_{i_0} = 2$ and $\pi_{i_0} = \mathrm{Sp}_2(\chi)$ for some unramified character χ of $F_w^\times$.

Thus if π is an irreducible smooth representation of $GL_n(F_w)$ with a $U_1(w)$ fixed vector but no $GL_n(\mathcal{O}_{F_w})$ fixed vector then

1. either $\pi = \chi_1 \boxplus ... \boxplus \chi_n$ with χ_i an unramified character of $F_w^\times$ for $i = 1, ..., n-1$ and with χ_n a character of $F_w^\times$ with conductor 1,
2. or $\pi = \chi_1 \boxplus ... \boxplus \chi_{n-2} \boxplus \mathrm{Sp}_2(\chi_{n-1})$ with χ_i an unramified character of $F_w^\times$ for $i = 1, ..., n-1$.

Consider first the first of these two cases. Let $\pi' = \chi_1 \boxplus ... \boxplus \chi_{n-1}$, an unramified representation of $GL_{n-1}(F_w)$. Also let $P \supset B_n$ denote the parabolic corresponding to the partition $n = (n-1)+1$. As $(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{U_1(w)}$ and $(\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi' \otimes \chi_n)^{U_1(w)}$ are one dimensional we must have

$$\begin{aligned}\pi^{U_1(w)} &= (\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{U_1(w)} \\ &= (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi' \otimes \chi_n)^{U_1(w)} \\ &= (\pi')^{GL_{n-1}(\mathcal{O}_{F_w})} \otimes \chi_n.\end{aligned}$$

From the last lemma we see that $V_\pi = \chi_n| \;|^{(1-n)/2}$ and that $U_w^{(j)}$ acts as $q_w^{j/2} T_w^{(j)} \otimes 1$. In particular π has no $U_0(w)$ fixed vector. Because

$$r_l(\pi' \boxplus \chi_n)^\vee(1-n) = r_l(\pi')^\vee(2-n)|\mathrm{Art}\,_{F_w}^{-1}|^{-1/2} \oplus (V_\pi \circ \mathrm{Art}\,_{F_w}^{-1})$$

the lemma follows.

Consider now the second of our two cases. Let $\pi' = \chi_1 \boxplus ... \boxplus \chi_{n-2}$, an unramified representation of $GL_{n-2}(F_w)$. Also let $P \supset B_n$ (resp. $P' \supset B_n$) denote the parabolic corresponding to the partition $n = (n-2)+2$ (resp. $n = 1 + ... + 1 + 2$). Because $\dim(\text{n-Ind}\,_{P'(F_w)}^{GL_n(F_w)} \chi_1 \otimes ... \otimes \chi_{n-2} \otimes \mathrm{Sp}_2(\chi_n))^{U_1(w)} = 1$

and $\dim(\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}\pi' \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} = 1$ we must have

$$\begin{aligned}\pi^{U_1(w)} &= (\text{n-Ind}\,_{P'(F_w)}^{GL_n(F_w)}\chi_1 \otimes \ldots \otimes \chi_{n-2} \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} \\ &= (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}\pi' \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} \\ &= (\pi')^{GL_{n-2}(\mathcal{O}_{F_w})} \otimes \mathrm{Sp}\,_2(\chi_n)^{U_1(w)}.\end{aligned}$$

Moreover V_α acts as $|\alpha|^{(2-n)/2}(1 \otimes V_\alpha)$ and $U_w^{(j)}$ acts as

$$q_w^j(T_w^{(j)} \otimes 1) + q_w^{n/2-1}(T_w^{(j-1)} \otimes U_w^{(1)}).$$

The induced representation $\text{n-Ind}\,_{B_2(F_w)}^{GL_2(F_w)}(\chi_n, \chi_n|\ |)$ has two irreducible constituents $(\chi_n|\ |^{1/2}) \circ \det$ and $\mathrm{Sp}\,_2(\chi_n)$. On $\text{n-Ind}\,_{B_2(F_w)}^{GL_2(F_w)}(\chi_n, \chi_n|\ |)^{U_1(w)}$ we have

$$V_\alpha = \begin{pmatrix} |\alpha|^{1/2}\chi_n(\alpha) & * \\ 0 & |\alpha|^{-1/2}\chi_n(\alpha) \end{pmatrix}$$

and

$$U_w^{(1)} = \begin{pmatrix} q_w^{1/2}\chi_n(\varpi_w) & 0 \\ * & q_w^{-1/2}\chi_n(\varpi_w) \end{pmatrix}.$$

On $(\chi_n|\ |^{1/2}) \circ \det$ we have

$$V_\alpha = |\alpha|^{1/2}\chi_n(\alpha)$$

and

$$U_w^{(1)} = q_w^{1/2}\chi_n(\varpi_w).$$

Thus on $\mathrm{Sp}\,_2(\chi_n)^{U_1(w)}$ we have

$$V_\alpha = |\alpha|^{-1/2}\chi_n(\alpha)$$

and

$$U_w^{(1)} = q_w^{-1/2}\chi_n(\varpi_w).$$

Hence on $\pi^{U_1(w)}$ we have

$$V_\alpha = |\alpha|^{(1-n)/2}\chi_n(\alpha)$$

and

$$U_w^{(j)} = q_w^j(T_w^{(j)} \otimes 1) + q_w^{(n-3)/2}(T_w^{(j-1)} \otimes \chi_n(\varpi_w)).$$

On the other hand

$$\begin{aligned}&(0) \to (r_l(\pi')^\vee(3-n)|\mathrm{Art}\,_{F_w}^{-1}|^{-1} \oplus (\chi_n|\ |^{(3-n)/2}) \circ \mathrm{Art}\,_{F_w}^{-1}) \to \\ &\to r_l(\pi' \boxplus \mathrm{Sp}\,_2(\chi_n))^\vee(1-n) \to (\chi_n|\ |^{(1-n)/2}) \circ \mathrm{Art}\,_{F_w}^{-1} \to (0).\end{aligned}$$

This is a short exact sequence of the desired form and $s(\mathrm{Frob}_w)$ has characteristic polynomial $(X - q_w^{(n-3)/2}\chi_n(\varpi_w))$ times

$$X^{n-2} - q_w t^{(1)} X^{n-3} + ... + (-1)^j q_w^{j+j(j-1)} t^{(j)} X^{n-2-j} + ... + (-1)^n q_w^{n^2-4n+4} t^{(n-2)},$$

where $t^{(j)}$ is the eigenvalue of $T_w^{(j)}$ on $(\pi')^{GL_{n-2}(\mathcal{O}_{F_w})}$. From the above formula for the $U_w^{(j)}$'s, we see that this product equals $P_\pi^{\mathrm{nr}}(X)$ and the lemma follows. □

Lemma 3.1.6 *Let π be an irreducible smooth representation of $GL_n(F_w)$ over $\overline{K}$.*

1. If $\pi^{\mathrm{Iw}_1(w)} \neq (0)$ then $r_l(\pi)^\vee(1-n)^{\mathrm{ss}}$ is a direct sum of one dimensional representations.

2. Suppose

$$\chi = (\chi_1, ..., \chi_n) : (k(w)^\times)^n \to \overline{K}^\times.$$

If $\pi^{\mathrm{Iw}_0(w),\chi} \neq (0)$ then

$$r(\pi)^\vee(1-n)|_{I_{F_w}}^{\mathrm{ss}} = (\chi_1 \circ \mathrm{Art}_{F_w}^{-1}) \oplus ... \oplus (\chi_n \circ \mathrm{Art}_{F_w}^{-1}).$$

(Here we think of χ_i as a character of $\mathcal{O}_{F_w}^\times \twoheadrightarrow k(w)^\times$.) Moreover if $\chi_i \neq \chi_j$ whenever $i \neq j$ then $r(\pi)^\vee(1-n)|_{I_{F_w}}$ is semisimple.

Proof: The key point is that $\pi^{\mathrm{Iw}_1(w)} \neq (0)$ if and only if π is a subquotient of a principal series representation $\text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1', ..., \chi_n')$ with each χ_i' tamely ramified. More precisely $\pi^{\mathrm{Iw}_0(w),\chi} \neq (0)$ if and only if π is a subquotient of a principal series representation $\text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1', ..., \chi_n')$ with each $\chi_i'|_{\mathcal{O}_{F_w}^\times} = \chi_i$. (See theorem 7.7 of [Ro]. In section 4 of that article some restrictions were placed on the characteristic of $\mathcal{O}_{F_w}/\wp_w$. However it is explained in remark 4.14 how these restrictions can be avoided in the case of GL_n. More precisely it is explained how to avoid these restrictions in the proof of theorem 6.3. The proof of theorem 7.7 relies only on lemma 3.6 and, via lemma 7.6, on lemma 6.2 and theorem 6.3. Lemmas 3.6 and 6.2 have no restrictions on the characteristic.) □

3.2. *GL_n over a local field: finite characteristic theory..* — We will keep the notation and assumptions of the last section. Let $l \not| q_w$ be a rational prime, K a finite extension of the field of fractions of the Witt vectors of an algebraic extension of $\mathbf{F}_l$, $\mathcal{O}$ the ring of integers of K, λ the maximal ideal of $\mathcal{O}$ and $k = \mathcal{O}/\lambda$.

Lemma 3.2.1 *Suppose that $l > n$ and $l|(q_w - 1)$. Suppose also that π is an unramified irreducible smooth representation of $GL_n(F_w)$ over $\overline{\mathbf{F}}_l$. Then* $\dim \pi^{GL_n(\mathcal{O}_{F_w})} = 1$. *Let $t_\pi^{(j)}$ denote the eigenvalue of $T_w^{(j)}$ on $\pi^{GL_n(\mathcal{O}_{F_w})}$. Set*

$$P_\pi(X) = X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}.$$

(Of course in $\overline{\mathbf{F}}_l$ we have $q_w = 1$ so we could have dropped it from this definition.) Suppose that $P_\pi(X) = (X-a)^m Q(X)$ with $m > 0$ and $Q(a) \neq 0$. Then

$$Q(V_{\varpi_w}) \pi^{GL_n(\mathcal{O}_{F_w})} \neq (0).$$

(Considered in $\pi^{U_0(w)}$.)

Proof: According to assertion VI.3 of [V2] we can find a partition $n = n_1 + ... + n_r$ corresponding to a parabolic $P \supset B_n$ and distinct, unramified characters $\chi_1, ..., \chi_r : F_w^\times \to \overline{\mathbf{F}}_l^\times$ such that $\pi = \text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\chi_1 \circ \det, ..., \chi_r \circ \det)$. Then

$$P_\pi(X) = \prod_{i=1}^r (X - \chi_i(\varpi_w))^{n_i}.$$

Suppose without loss of generality that $a = \chi_1(\varpi_w)$.

For $i = 1, ..., r$ set $w_i' = w_{n, n_1 + ... + n_i}$ (in the notation established in the fourth paragraph of section 3.1). Then $\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\chi_1 \circ \det, ..., \chi_r \circ \det)^{U_0(w)}$ has a basis consisting of functions φ_i for $i = 1, ..., r$, where the support of φ_i is $P(F_w) w_i' U_0(w)$ and $\varphi_i(w_i') = 1$. Note that

$$\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\chi_1 \circ \det, ..., \chi_r \circ \det)^{GL_n(\mathcal{O}_{F_w})}$$

is spanned by $\varphi_1 + ... + \varphi_r$.

We have

$$V_{\varpi_w} \varphi_i = \sum_j (V_{\varpi_w} \varphi_i)(w_j') \varphi_j.$$

But, as in the proof of lemma 3.1.3, we also have

$$(V_{\varpi_w} \varphi_i)(w_j') = \chi_j(\varpi_w) \sum_{x \in X^{n_1 + ... + n_j - 1}} \varphi_i \begin{pmatrix} 1_{n_1 + ... + n_j - 1} & 0 & 0 \\ x & 0 & 1 \\ 0 & w_{n_{j+1} + ... + n_r} & 0 \end{pmatrix}.$$

A matrix $g \in GL_n(\mathcal{O}_{F_w})$ lies in $P(\mathcal{O}_{F_w}) w_i' U_0(w)$ if and only if i is the largest integer such that $(0, ..., 0, 1)$ lies in the $k(w)$ span of the reduction modulo $\wp_w$ of the last $n_i + ... + n_r$ rows of g. Thus

$$(V_{\varpi_w} \varphi_i)(w_j')$$

is

- 0 if $i > j$,
- $q_w^{n_i-1}\chi_j(\varpi_w) = \chi_j(\varpi_w)$ if $i = j$, and
- $(q_w^{n_i} - 1)q_w^{n_{i+1}+...+n_j-1}\chi_j(\varpi_w) = 0$ if $i < j$.

Thus, for $i = 1, ..., r$, we have

$$V_{\varpi_w}\varphi_i = \chi_i(\varpi_w)\varphi_i$$

and

$$Q(V_{\varpi_w})(\varphi_1 + ... + \varphi_r) = Q(\chi_1(\varpi_w))\varphi_1$$

and the lemma follows. □

Lemma 3.2.2 *Suppose that $l > n$ and $l|(q_w - 1)$. Let R be a complete local $\mathcal{O}$-algebra. Let M be an R-module with a smooth action of $GL_n(F_w)$ such that for all open compact subgroups $U \subset GL_n(F_w)$ the module of invariants M^U is finite and free over $\mathcal{O}$. Suppose also that for $j = 1, .., n$ there are elements $t_j \in R$ with $T_w^{(j)} = t_j$ on $M^{GL_n(\mathcal{O}_{F_w})}$. Set*

$$P(X) = X^n + \sum_{j=1}^{n}(-1)^j q_w^{j(j-1)/2} t_j X^{n-j} \in R[X].$$

Suppose that in $R[X]$ we have a factorisation $P(X) = (X - a)Q(X)$ with $Q(a) \in R^\times$. Suppose finally that $M \otimes_{\mathcal{O}} \overline{K}$ is semi-simple over the ring $(R \otimes_{\mathcal{O}} \overline{K})[GL_n(F_w)]$ and that, if π is an R-invariant irreducible $GL_n(F_w)$-constituent of $M \otimes_{\mathcal{O}} \overline{K}$ with a $U_0(w)$-fixed vector, then either π is unramified or

$$\begin{aligned}P(X) = (X - V_{\varpi_w})(X^{n-1} - U_w^{(1)}X^{n-2} + ... + (-1)^j q_w^{j(j-1)/2} U_w^{(j)} X^{n-1-j} +\\ + ... + (-1)^n q_w^{(n-1)(n-2)/2} U_w^{(n-1)})\end{aligned}$$

on $\pi^{U_0(w)}$ (i.e. for $j = 1, ..., n$ the coefficient of X^{n-j} on the right hand side acts on the one dimensional space $\pi^{U_0(w)}$ by $(-1)^j q_w^{j(j-1)/2} t_j$). Then $Q(V_\varpi)$ gives an isomorphism

$$Q(V_{\varpi_w}) : M^{GL_n(\mathcal{O}_{F_w})} \xrightarrow{\sim} M^{U_0(w), V_{\varpi_w} = a}.$$

Proof: Lemma 3.1.3 tells us that

$$Q(V_{\varpi_w}) : M^{GL_n(\mathcal{O}_{F_w})} \longrightarrow M^{U_0(w), V_{\varpi_w} = a}.$$

Let π be an R-invariant irreducible $GL_n(F_w)$-constituent of $M \otimes_{\mathcal{O}} \overline{K}$ with $\pi^{U_0(w), V_{\varpi_w} = a} \neq (0)$. If π is ramified then lemma 3.1.5 tells us that

$$\begin{aligned}(q_w^{-1}a)^{n-1} - U_w^{(1)}(q_w^{-1}a)^{n-2} ... + (-1)^j q_w^{j(j-1)/2} U_w^{(j)} (q_w^{-1}a)^{n-1-j} +\\ + ... + (-1)^n q_w^{(n-1)(n-2)/2} U_w^{(n-1)} = 0\end{aligned}$$

on $\pi^{U_0(w)}$. Thus $Q(a) \in \mathfrak{m}_R$, which contradicts our hypothesis. Thus π is unramified. By lemma 3.1.3 and the assumption that a is a simple root of $P(X)$, we see that $\dim \pi^{U_0(w), V_{\varpi_w}=a} \leq 1 = \dim \pi^{GL_n(\mathcal{O}_{F_w})}$. Thus

$$\dim(M \otimes_{\mathcal{O}} \overline{K})^{U_0(w), V_{\varpi_w}=a} \leq \dim(M \otimes_{\mathcal{O}} \overline{K})^{GL_n(\mathcal{O}_{F_w})}.$$

Hence it suffices to show that $Q(V_{\varphi_w}) \otimes \overline{k}$ is injective. Suppose not. Choose a non-zero vector $x \in \ker(Q(V_{\varphi_w}) \otimes \overline{k})$ such that $\mathfrak{m}_R x = (0)$. Let N' denote the $\overline{k}[GL_n(F_w)]$-submodule of $M \otimes_{\mathcal{O}} \overline{k}$ generated by x. Let N denote an irreducible quotient of N'. Then by lemma 3.2.1

$$Q(V_{\varpi_w}) N^{GL_n(\mathcal{O}_{F_w})} \neq (0),$$

a contradiction and the lemma is proved. $\square$

3.3. *Automorphic forms on unitary groups..* — Fix a positive integer $n \geq 2$ and a prime $l > n$.

Fix an imaginary quadratic field E in which l splits and a totally real field F^+. Set $F = F^+E$. Fix a finite non-empty set of places $S(B)$ of places of F^+ with the following properties:

- Every element of $S(B)$ splits in F.
- $S(B)$ contains no place above l.
- If n is even then

$$n[F^+ : \mathbf{Q}]/2 + \#S(B) \equiv 0 \bmod 2.$$

Choose a division algebra B with centre F with the following properties:

- $\dim_F B = n^2$.
- $B^{\mathrm{op}} \cong B \otimes_{E,c} E$.
- B splits outside $S(B)$.
- If w is a prime of F above an element of $S(B)$, then B_w is a division algebra.

If $\ddagger$ is an involution on B with $\ddagger|_F = c$ then we can define a reductive algebraic group $G_\ddagger/F^+$ by setting

$$G_\ddagger(R) = \{g \in B \otimes_{F^+} R : \ g^{\ddagger \otimes 1} g = 1\}$$

for any F^+-algebra R. Fix an involution $\ddagger$ on B such that

- $\ddagger|_F = c$,
- for a place $v|\infty$ of F^+ we have $G_\ddagger(F_v^+) \cong U(n)$, and
- for a finite place $v \notin S(B)$ of F^+ the group $G_\ddagger(F_v^+)$ is quasi-split.

Because either n is odd or

$$n[F^+ : \mathbf{Q}]/2 + \#S(B) \equiv 0 \bmod 2,$$

this is always possible. (The argument is exactly analogous to the proof of lemma 1.7.1 of [HT].) From now on we will write G for $G_\ddagger$.

We can choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\ddagger = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is maximal for all primes w of F which are split over F^+. (Start with any order. Replacing it by its intersection with its image under $\ddagger$ gives an order $\mathcal{O}'_B$ with $(\mathcal{O}'_B)^\ddagger = \mathcal{O}'_B$. For all but finitely many primes v of F^+ the completion $\mathcal{O}'_{B,v}$ will be a maximal order in B_v. Let R denote the finite set of primes which split in F and for which $\mathcal{O}'_{B,v}$ is not maximal. For $v \in R$ choose a maximal order $\mathcal{O}''_{B,v}$ of B_v with $(\mathcal{O}''_{B,v})^\ddagger = \mathcal{O}''_{B,v}$ (e.g. $\mathcal{O}_{B,w} \oplus \mathcal{O}_{B,w}^\ddagger$ where w is a prime of F above v and $\mathcal{O}_{B,w}$ is a maximal order in B_w). Let $\mathcal{O}_B$ be the unique order with $\mathcal{O}_{B,v} = \mathcal{O}''_{B,v}$ if $v \in R$ and $\mathcal{O}_{B,v} = \mathcal{O}'_{B,v}$ otherwise.) This choice gives a model of G over $\mathcal{O}_{F^+}$. (This model may be very bad at primes v which do not split in F, but this will not concern us.)

Let v be a place of F^+ which splits in F. If $v \notin S(B)$ choose an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F_v})$ such that $i_v(x^\ddagger) = {}^t i_v(x)^c$. The choice of a prime w of F above v then gives us an identification

$$\begin{aligned} i_w : G(F_v^+) &\xrightarrow{\sim} GL_n(F_w) \\ i_v^{-1}(x, {}^t x^{-c}) &\longmapsto x \end{aligned}$$

with $i_w G(\mathcal{O}_{F^+,v}) = GL_n(\mathcal{O}_{F,w})$ and $i_{w^c} = {}^t(c \circ i_w)^{-1}$. If $v \in S(B)$ and w is a prime of F above v we get an isomorphism

$$i_w : G(F_v^+) \xrightarrow{\sim} B_w^\times$$

with $i_w G(\mathcal{O}_{F^+,v}) = \mathcal{O}_{B,w}^\times$ and $i_{w^c}(x) = (i_w(x)^\ddagger)^{-1}$.

Let S_l denote the primes of F^+ above l. Suppose that R is a finite set of primes of F^+ which split in F such that R is disjoint from $S_l \cup S(B)$. Let $T \supset S_l \cup R \cup S(B)$ denote a finite set of primes of F^+ which split in F. Fix a set $\widetilde{T}$ of primes of F such that $\widetilde{T} \coprod {}^c\widetilde{T}$ is the set of all primes of F above T. If $S \subset T$ write $\widetilde{S}$ for the preimage of S in $\widetilde{T}$. If $v \in T$ we will write $\widetilde{v}$ for the element of $\widetilde{T}$ above v. Write S_∞ for the set of infinite places of F^+.

Let k be an algebraic extension of $\mathbf{F}_l$ and K a finite, totally ramified extension of the fraction field of the Witt vectors of k such that K contains the image of every embedding $F \hookrightarrow \overline{K}$. Let $\mathcal{O}$ denote the ring of integers of K and let λ denote its maximal ideal. Let I_l denote the set of embeddings $F^+ \hookrightarrow K$, so that there is a natural surjection $I_l \twoheadrightarrow S_l$. Let $\widetilde{I}_l$ denote the set of embeddings $F \hookrightarrow K$ which give rise to a prime of $\widetilde{S}_l$. Thus there is a natural bijection $\widetilde{I}_l \xrightarrow{\sim} I_l$.

For an n-tuple of integers $a = (a_1, ..., a_n)$ with $a_1 \geq ... \geq a_n$ there is an irreducible representation defined over $\mathbf{Q}$:

$$\xi_a : GL_n \longrightarrow GL(W_a)$$

with highest weight

$$\mathrm{diag}(t_1, ..., t_n) \longmapsto \prod_{i=1}^{n} t_i^{a_i}.$$

(N.B. This is not the same convention used in [HT].) We can choose a model

$$\xi_a : GL_n \longrightarrow GL(M_a)$$

of ξ_a over $\mathbf{Z}$. (So M_a is a $\mathbf{Z}$-lattice in W_a.)

Let Wt_n denote the subset of $(\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbf{Q}_l})}$ consisting of elements a which satisfy

– $a_{\tau c,i} = -a_{\tau,n+1-i}$ and
– $a_{\tau,1} \geq ... \geq a_{\tau,n}$.

If $a \in \mathrm{Wt}_n$ then we get a K-vector space W_a and irreducible representation

$$\begin{aligned} \xi_a : G(F_l^+) &\longrightarrow GL(W_a) \\ g &\longmapsto \otimes_{\tau \in \widetilde{I}_l} \xi_{a_\tau}(\tau i_\tau g). \end{aligned}$$

The representation ξ_a contains a $G(\mathcal{O}_{F^+,l})$-invariant $\mathcal{O}$-lattice M_a.

For $v \in S(B)$, let $\rho_v : G(F_v^+) \to GL(M_{\rho_v})$ denote a representation of $G(F_v^+)$ on a finite free $\mathcal{O}$-module such that ρ_v has open kernel and $M_{\rho_v} \otimes_{\mathcal{O}} \overline{K}$ is irreducible. If $\mathrm{JL}\,(\rho_v \circ i_{\widetilde{v}}^{-1}) = \mathrm{Sp}_{m_v}(\pi_{\widetilde{v}})$ then set

$$\widetilde{r}_{\widetilde{v}} = r_l(\pi_{\widetilde{v}}|\ \ |^{(n/m_v-2)(1-m_v)/2}).$$

We will *suppose that*

$$\widetilde{r}_{\widetilde{v}} : \mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}) \longrightarrow GL_{n/m_v}(\mathcal{O}).$$

(This is a condition on K. A priori this representation is into $GL_{n/m_v}(\overline{K})$, but if K is sufficiently large it can be replaced by a conjugate valued in $GL_{n/m_v}(\mathcal{O})$. Because $\widetilde{r}_{\widetilde{v}}$ is absolutely irreducible it suffices to check that $\det \widetilde{r}_{\widetilde{v}}$ takes unit values, and this follows because v does not lie above l and because the central character of ρ_v takes unit values.)

For $v \in R$ let $U_{0,v}$ be an open compact subgroup of $G(F_v^+)$ and let

$$\chi_v : U_{0,v} \longrightarrow \mathcal{O}^\times$$

be a homomorphism with open kernel.

We will call an open compact subgroup $U \subset G(\mathbf{A}_{F^+}^\infty)$ *sufficiently small* if for some place v its projection to $G(F_v^+)$ contains only one element of finite order, namely 1.

Let A denote an $\mathcal{O}$-algebra.

Suppose that U is an open compact subgroup of $G(\mathbf{A}_{F^+}^\infty)$ for which the projection to $G(F_v^+)$ is contained in $U_{0,v}$ for all $v \in R$. Suppose also that $a \in \mathrm{Wt}_n$ and that for $v \in S(B)$, ρ_v is as in the last paragraph but two. Set

$$M_{a,\{\rho_v\},\{\chi_v\}} = M_a \otimes \left(\bigotimes_{v \in S(B)} M_{\rho_v} \right) \otimes \left(\bigotimes_{v \in R} \mathcal{O}(\chi_v) \right).$$

Suppose that either A is a K-algebra or that the projection of U to $G(F_l^+)$ is contained in $G(\mathcal{O}_{F^+,l})$. Then we define a space of automorphic forms

$$S_{a,\{\rho_v\},\{\chi_v\}}(U, A)$$

to be the space of functions

$$f : G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty) \longrightarrow A \otimes_{\mathcal{O}} M_{a,\{\rho_v\},\{\chi_v\}}$$

such that

$$f(gu) = u_{S_l \cup S(B) \cup R}^{-1} f(g)$$

for all $u \in U$ and $g \in G(\mathbf{A}_{F^+}^\infty)$. Here $u_{S_l \cup S(B) \cup R}$ denotes the projection of u to $\prod_{v \in S_l \cup S(B) \cup R} G(F_v^+)$. If V is any compact subgroup of $G(\mathbf{A}_{F^+}^\infty)$ for which the projection to $G(F_v^+)$ is contained in $U_{0,v}$ for all $v \in R$, then we define $S_{a,\{\rho_v\},\{\chi_v\}}(V, A)$ to be the union of the $S_{a,\{\rho_v\},\{\chi_v\}}(U, A)$ as U runs over open compact subgroups containing V which have projection to $G(F_v^+)$ is contained in $U_{0,v}$ for all $v \in R$.

If $g \in G(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v \in R} U_{0,v}$ (and either A is a K-algebra or $g_l \in G(\mathcal{O}_{F^+,l})$) and if $V \subset gUg^{-1}$ then there is a natural map

$$g : S_{a,\{\rho_v\},\{\chi_v\}}(U, A) \longrightarrow S_{a,\{\rho_v\},\{\chi_v\}}(V, A)$$

defined by

$$(gf)(h) = g_{S_l \cup S(B) \cup R} f(hg).$$

We see that if V is a normal subgroup of U then

$$S_{a,\{\rho_v\},\{\chi_v\}}(U, A) = S_{a,\{\rho_v\},\{\chi_v\}}(V, A)^U.$$

If U is open then the A-module $S_{a,\{\rho_v\}}(U, A)$ is finitely generated. If U is open and sufficiently small then it is free of rank $\# G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty)/U$. If A is flat over $\mathcal{O}$ or if U is sufficiently small then

$$S_{a,\{\rho_v\},\{\chi_v\}}(U, A) = S_{a,\{\rho_v\},\{\chi_v\}}(U, \mathcal{O}) \otimes_{\mathcal{O}} A.$$

Suppose that U_1 and U_2 are compact subgroups whose projections to $G(F_v^+)$ are contained in $U_{0,v}$ for all $v \in R$ and that $g \in G(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v \in R} U_{0,v}$. If A is not a K-algebra suppose that $g_l \in G(\mathcal{O}_{F^+,l})$ and that $u_l \in G(\mathcal{O}_{F^+,l})$ for all $u \in U_1 \cup U_2$. Suppose also that $\# U_1 g U_2 / U_2 < \infty$. (This will be automatic if U_1 and U_2 are open.) Then we define a linear map

$$[U_1 g U_2] : S_{a,\{\rho_v\},\{\chi_v\}}(U_2, A) \longrightarrow S_{a,\{\rho_v\},\{\chi_v\}}(U_1, A)$$

by

$$([U_1 g U_2] f)(h) = \sum_i (g_i)_{S_l \cup S(B) \cup R} f(h g_i)$$

if $U_1 g U_2 = \coprod_i g_i U_2$.

Lemma 3.3.1 *Let $U \subset G(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v \in R} U_{0,v}$ be a sufficiently small open compact subgroup and let $V \subset U$ be a normal open subgroup. Let A be an $\mathcal{O}$-algebra. Suppose that either A is a K-algebra or the projection of U to $G(F_l^+)$ is contained in $G(\mathcal{O}_{F^+,l})$. Then $S_{a,\{\rho_v\},\{\chi_v\}}(V, A)$ is a finite free $A[U/V]$-module and $\mathrm{tr}_{U/V}$ gives an isomorphism from the coinvariants $S_{a,\{\rho_v\},\{\chi_v\}}(V, A)_{U/V}$ to $S_{a,\{\rho_v\},\{\chi_v\}}(U, A)$.*

Proof: Suppose that

$$G(\mathbf{A}_{F^+}^{\infty}) = \coprod_{j \in J} G(F^+) g_j U.$$

For all $j \in J$ we have $g_j^{-1} G(F^+) g_j \cap U = \{1\}$. (Because this intersection is finite and U is sufficiently small.) Thus

$$G(\mathbf{A}_{F^+}^{\infty}) = \coprod_{j \in J} \coprod_{u \in U/V} G(F^+) g_j u V.$$

Moreover

$$\begin{array}{rcl} S_{a,\{\rho_v\},\{\chi_v\}}(U, A) & \xrightarrow{\sim} & \bigoplus_{j \in J} M_{a,\{\rho_v\},\{\chi_v\}} \otimes_{\mathcal{O}} A \\ f & \longmapsto & (f(g_j))_j \end{array}$$

and

$$\begin{array}{rcl} S_{a,\{\rho_v\},\{\chi_v\}}(V, A) & \xrightarrow{\sim} & \bigoplus_{j \in J} \bigoplus_{u \in U/V} M_{a,\{\rho_v\},\{\chi_v\}} \otimes_{\mathcal{O}} A \\ f & \longmapsto & (f(g_j u))_{j,u}. \end{array}$$

Alternatively we get an isomorphism of $A[U/V]$-modules

$$\begin{array}{rcl} S_{a,\{\rho_v\},\{\chi_v\}}(V, A) & \xrightarrow{\sim} & \bigoplus_{j \in J} M_{a,\{\rho_v\},\{\chi_v\}} \otimes_{\mathcal{O}} A[U/V] \\ f & \longmapsto & (\sum_{u \in U/V} u_{S_l \cup R \cup S(B)} f(g_j u) \otimes u^{-1})_j. \end{array}$$

Then

$$\begin{array}{rcl} S_{a,\{\rho_v\},\{\chi_v\}}(V,A)_{U/V} & \stackrel{\sim}{\longrightarrow} & \bigoplus_{j\in J} M_{a,\{\rho_v\},\{\chi_v\}} \otimes_{\mathcal{O}} A \\ f & \longmapsto & (\sum_{u\in U/V} u_{S_l\cup R\cup S(B)} f(g_j u))_j. \end{array}$$

In fact we have a commutative diagram

$$\begin{array}{ccc} S_{a,\{\rho_v\},\{\chi_v\}}(V,A)_{U/V} & \stackrel{\mathrm{tr}_{U/V}}{\longrightarrow} & S_{a,\{\rho_v\},\{\chi_v\}}(U,A) \\ \downarrow & & \downarrow \\ \bigoplus_{j\in J} M_{a,\{\rho_v\},\{\chi_v\}} \otimes_{\mathcal{O}} A & = & \bigoplus_{j\in J} M_{a,\{\rho_v\},\{\chi_v\}} \otimes_{\mathcal{O}} A \end{array}$$

where the vertical maps are the above isomorphisms. The lemma follows. $\Box$

Proposition 3.3.2 *Fix* $\iota : K \hookrightarrow \mathbf{C}$.

1. $S_{a,\{\rho_v\},\emptyset}(\{1\},\mathbf{C})$ *is a semi-simple admissible* $G(\mathbf{A}_{F^+}^{\infty})$*-module.*
2. *If* $S(B) \neq \emptyset$ *and* $\pi = \otimes_v \pi_v$ *is an irreducible constituent of the space* $S_{a,\{\rho_v\},\emptyset}(\{1\},\mathbf{C})$ *then there is an automorphic representation* $\mathrm{BC}_{\iota}(\pi)$ *of* $(B\otimes \mathbf{A})^{\times}$ *with the following properties.*
   - $\mathrm{BC}_{\iota}(\pi)\circ(-\ddagger) = \mathrm{BC}_{\iota}(\pi)$.
   - *If a prime* v *of* F^+ *splits as* ww^c *in* F *then* $\mathrm{BC}_{\iota}(\pi)_w \cong \pi_v \circ i_w^{-1}$.
   - *If* v *is an infinite place of* F^+ *and* $\tau : F \hookrightarrow \mathbf{C}$ *lies above* v *then* $\mathrm{BC}_{\iota}(\pi)_v$ *is cohomological for* $(\xi_{a_{\iota^{-1}\tau}}\circ\tau)\otimes(\xi_{a_{\iota^{-1}\tau c}}\circ\tau c)$.
   - *If* v *is a prime of* F^+ *which is unramified, inert in* F *and if* π_v *has a fixed vector for a hyperspecial maximal compact subgroup of* $G(F_v)$ *then* $\mathrm{BC}_{\iota}(\pi)_v$ *has a* $GL_n(\mathcal{O}_{F,v})$*-fixed vector.*
   - *If* $v \in S(B)$ *and* π_v *has a* $G(\mathcal{O}_{F,v})$ *fixed vector and* w *is a prime of* F *above* v *then* $\mathrm{BC}_{\iota}(\pi)_w$ *is an unramified twist of* $(\iota\rho_v^{\vee})\circ i_w^{-1}$.
3. *If* $S(B) \neq \emptyset$ *and* $\pi = \otimes_v \pi_v$ *is an irreducible constituent of the space* $S_{a,\{\rho_v\},\emptyset}(\{1\},\mathbf{C})$ *such that for* $v\in S(B)$ *the representation* π_v *has a* $G(\mathcal{O}_{F^+,v})$*-fixed vector, then there is a positive integer* $m|n$ *and there is a cuspidal automorphic representation* Π *of* $GL_{n/m}(\mathbf{A}_F)$ *with the following properties.*
   - $\Pi^{\vee}\circ c = \Pi|\ |^{m-1}$.
   - *If a prime* $v \notin S(B)$ *of* F^+ *splits as* ww^c *in* F *then* $\Pi_w \boxplus \Pi_w|\ |\boxplus \ldots \boxplus \Pi_w|\ |^{m-1} \cong \pi_v \circ i_w^{-1}$.
   - *If* v *is an infinite place of* F^+ *and* $\tau : F \hookrightarrow \mathbf{C}$ *lies above* v *then* $\Pi_v|\ |^{n(m-1)/(2m)}$ *is cohomological for* $(\xi_{b_\tau}\circ\tau)\otimes(\xi_{b_{\tau c}}\circ\tau c)$ *and* $b_{\tau,i} = a_{\tau,m(i-1)+j} + (m-1)(i-1)$ *for every* $j=1,\ldots,m$.
   - *If* v *is a prime of* F^+ *which is unramified, inert in* F *and if* π_v *has a fixed vector for a hyperspecial maximal compact subgroup of* $G(F_v)$ *then* Π_v *has a* $GL_{n/m}(\mathcal{O}_{F,v})$*-fixed vector.*
   - *If* $m>1$ *and* w *is a prime of* F *above a prime* $v\in S(B)$ *then* Π_w *is cuspidal.*

– *If* $v \in S(B)$ *and* w *is a prime of* F *above* v *then* $\mathrm{JL}\,(\iota\rho_v \circ i_w^{-1})^\vee$ *is an unramified twist of* $\mathrm{Sp}_m(\Pi_w)$. *(In the case* $m = 1$ *and* Π_w *is not cuspidal we interpret* $\mathrm{Sp}_m(\Pi_w)$ *as* Π_w*.)*

If for one place $v_0 \notin S(B)$ *of* F^+*, which splits in* F*, the representation* π_{v_0} *is generic, then for all places* $v \notin S(B)$ *of* F^+*, which split in* F*, the representation* π_v *is generic.*

4. Suppose Π *is a cuspidal automorphic representation of* $GL_n(\mathbf{A}_F)$ *with the following properties.*

– $\Pi^\vee \circ c = \Pi$.

– *If* v *is an infinite place of* F^+ *and* $\tau : F \hookrightarrow \mathbf{C}$ *lies above* v *then* Π_v *is cohomological for* $(\xi_{a_{\iota^{-1}\tau}} \circ \tau) \otimes (\xi_{a_{\iota^{-1}\tau c}} \circ \tau c)$.

– *If* $v \in S(B)$ *and* w *is a prime of* F *above* v *then* Π_w *is an unramified twist of* $\mathrm{JL}\,((\iota\rho_v^\vee) \circ i_w^{-1})$.

Then there is an irreducible constituent π *of* $S_{a,\{\rho_v\},\emptyset}(\{1\}, \mathbf{C})$ *with the following properties.*

– *For* $v \in S(B)$ *the representation* π_v *has a* $G(\mathcal{O}_{F^+,v})$*-fixed vector.*

– *If a prime* $v \notin S(B)$ *of* F^+ *splits as* ww^c *in* F *then* $\pi_v \cong \Pi_w \circ i_w$.

– *If* v *is a prime of* F^+ *which is inert and unramified in* F *and if* Π_w *is unramified then* π_v *has a fixed vector for a hyperspecial maximal compact subgroup of* $G(F_v)$.

Proof: If $\tau \in \widetilde{I}_l$ then $\iota\tau : F \to \mathbf{C}$ and hence $F_\infty \to \mathbf{C}$. Then $W_a \otimes_{K,\iota} \mathbf{C}$ is naturally a continuous $G(F_\infty^+)$-module:

$$g \longmapsto \otimes_{\tau \in \widetilde{I}_l} \xi_{a_\tau}(\iota\tau g).$$

Denote this action by $\xi_{a,\iota}$. Similarly $M_{a,\{\rho_v\},\emptyset} \otimes_{\mathcal{O},\iota} \mathbf{C}$ becomes a continuous $G(F_\infty^+) \times \prod_{v \in S(B)} G(F_v)$-module and hence (via projection) also a continuous $G(\mathbf{A}_{F^+})$-module, which we will denote $(M_{a,\{\rho_v\},\emptyset} \otimes_{\mathcal{O},\iota} \mathbf{C})_\infty$ to make clear which action is being considered. Let $\mathcal{A}$ denote the space of automorphic forms on $G(F^+)\backslash G(\mathbf{A}_{F^+})$. We have an isomorphism

$$i : S_{a,\{\rho_v\},\emptyset}(U, \mathbf{C}) \xrightarrow{\sim} \mathrm{Hom}_{U \times G(F_\infty^+)}((M_{a,\{\rho_v\},\emptyset} \otimes_{\mathcal{O},\iota} \mathbf{C})_\infty^\vee, \mathcal{A})$$

given by

$$i(f)(\alpha)(g) = \alpha(\xi_{a,\iota}(g_\infty)^{-1}(\xi_a(g_l) f(g^\infty))).$$

(We remark that the elements of $S_{a,\{\rho_v\},\emptyset}(U, \mathbf{C})$ are not continuous functions, because our definition of $S_{a,\{\rho_v\},\emptyset}(U, A)$ was designed to give continuous functions when A is a topological $\mathcal{O}$-algebra. The map ι makes $\mathbf{C}$ an $\mathcal{O}$-algebra, but is not continuous.)

The first part now becomes a standard fact. The second part follows from theorem A.5.2 of [CL], except that theorem A.5.2 of [CL] only identifies

$\mathrm{BC}_\iota(\pi)_v$ for all but finitely many v. We can easily adapt the argument to identify $\mathrm{BC}_\iota(\pi)_v$ at all split places, as is described in the proof of theorem VI.2.1 of [HT] (page 202). It is equally easy to control $\mathrm{BC}_\iota(\pi)_v$ at places where π_v has a fixed vector for a hyperspecial maximal compact subgroup. One just chooses the set S in the proof of theorem A.5.2 of [CL] not containing v. The third part follows from the second, theorem VI.1.1 of [HT] and the main result of [MW]. As for the fourth part, the existence of some descent (controlled at all but finitely many places) follows from theorem VI.1.1 of [HT] and the argument for proposition 2.3 of [Cl] as completed by theorem A.3.1 of [CL]. That this descent has all the stated properties follows from the earlier parts of this proposition. $\Box$

Corollary 3.3.3 *$S_{a,\{\rho_v\},\{\chi_v\}}(\{1\},K)$ is a semi-simple admissible $G(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v\in R} U_{0,v}$-module.*

Proof: This reduces to the case $R=\emptyset$ which follows from proposition 3.3.2. $\Box$

Combining the above proposition with theorem VII.1.9 of [HT] we obtain the following result.

Proposition 3.3.4 *Let $\overline{K}^0$ denote the algebraic closure of $\mathbf{Q}_l$ in $\overline{K}$. Suppose that $\pi = \otimes_{v\notin R}\pi_v$ is an irreducible constituent of $S_{a,\{\rho_v\},\{\chi_v\}}(\prod_{v\in R} U_{0,v}, \overline{K})$ then there is a continuous semi-simple representation*

$$r_\pi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{K}^0)$$

with the following properties.

1. *If $v \notin R\cup S(B)\cup S_l$ is a prime of F^+ which splits $v = ww^c$ in F, then*
$$r_\pi|_{G_{F_w}}^{\mathrm{ss}} = (r_l(\pi_w \circ i_w^{-1})^\vee(1-n))^{\mathrm{ss}}.$$
2. *$r_\pi^c \cong r_\pi^\vee \epsilon^{1-n}$.*
3. *If $v \in S(B)$ splits $v = ww^c$ in F then*
$$r_\pi|_{G_{F_w}}^{\mathrm{ss}} = (r_l(\mathrm{JL}\,(\pi_w \circ i_w^{-1}))^\vee(1-n))^{\mathrm{ss}}.$$
4. *If v is a prime of F^+ which is inert and unramified in F and if π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(F_v^+)$ then $r_\pi|_{W_{F_v}}$ is unramified.*
5. *If w is a prime of F above l then r_π is potentially semi-stable at w. If moreover $\pi_{w|_{F^+}}$ is unramified then r_π is crystalline at w.*

6. If $\tau : F \hookrightarrow K$ gives rise to a prime w of F then

$$\dim_{\overline{K}^0} \mathrm{gr}^i (r_\pi \otimes_{\tau, F_w} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_w/F_w)} = 0$$

unless $i = a_{\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

$$\dim_{\overline{K}^0} \mathrm{gr}^i (r_\pi \otimes_{\tau, F_w} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_w/F_w)} = 1.$$

7. If for some place $v \notin S(B) \cup R$ of F^+ which splits in F the representation π_v is not generic then r_π is reducible.

Proof: Let m and Π be as in part 3 of proposition 3.3.2. Let $S' \supset S_l$ be any finite set of finite places of F^+ which are unramified in F. Choose a character $\psi : \mathbf{A}_F^\times \to \mathbf{C}^\times$ such that

- $\psi^{-1} = \psi^c$;
- ψ is unramified above S'; and
- if $\tau : F \hookrightarrow \mathbf{C}$ gives rise to an infinite place v of F then

$$\psi_v : z \longmapsto (\tau z / |\tau z|)^{\delta_\tau}$$

where $|z|^2 = zz^c$ and $\delta_\tau = 0$ if either m or n/m is odd and $\delta_\tau = \pm 1$ otherwise.

The existence of such a character is proved as in the proof of lemma VII.2.8 of [HT]. Then

$$r_\pi = R_l(\Pi \otimes \psi| \ |^{(m-1)/2})^\vee (1-n) \otimes R_l(\psi^{-1}| \ |^{(n/m-1)(m-1)/2})^\vee \otimes (1 \oplus \epsilon^{-1} \oplus ... \oplus \epsilon^{1-m})$$

is independent of the choice of S' and ψ and satisfies the requirements of the proposition. (See theorem VII.1.9 of [HT]. We use the freedom to vary S' to verify property 4. Note that if $m = 1$ then we simply have $r_\pi = R_l(\Pi)^\vee (1-n)$.) $\Box$

3.4. *Unitary group Hecke algebras..* — Keep the notation and assumptions of the last section. Further suppose that $T \supset Q \cup R \cup S(B) \cup S_l$ is a finite set of places of F^+ and that

$$U = \prod_v U_v \subset G(\mathbf{A}_{F^+}^\infty)$$

is a sufficiently small open compact subgroup such that

1. if $v \notin T$ splits in F then $U_v = G(\mathcal{O}_{F^+,v})$,
2. if $v \in R$ then $U_v = i_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$,
3. and if $v \in Q$ then $U_v = i_{\widetilde{v}}^{-1} U_1(\widetilde{v})$.

If $v \in S(B)$ also suppose that the representation

$$\widetilde{r}_{\widetilde{v}} : G_{F_{\widetilde{v}}} \longrightarrow GL_{n/m_v}(\mathcal{O})$$

has the following properties:

1. $\widetilde{r}_{\widetilde{v}} \otimes k$ is absolutely irreducible,
2. every irreducible subquotient of $(\widetilde{r}_{\widetilde{v}} \otimes k)|_{I_{F_{\widetilde{v}}}}$ is absolutely irreducible,
3. and $\widetilde{r}_{\widetilde{v}} \otimes k \not\cong \widetilde{r}_{\widetilde{v}} \otimes k(i)$ for $i = 1, ..., m_v$.

By the first of these properties we see that the realisation over $\mathcal{O}$ we chose for $\widetilde{r}_{\widetilde{v}} \otimes \overline{K}$ is in fact unique up to equivalence. If $v \in R$ also suppose that χ_v is a character of $\mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v})$ and hence of the form

$$g \longmapsto \prod_{i=1}^{n} \chi_{v,i}(g_{ii})$$

where $\chi_{v,i} : k(\widetilde{v})^\times \to \mathcal{O}^\times$.

We will denote by

$$\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)$$

the $\mathcal{O}$-subalgebra of $\mathrm{End}\,(S_{a,\{\rho_v\},\{\chi_v\}}(U,\mathcal{O}))$ generated by the Hecke operators $T_w^{(j)}$ (or strictly speaking $i_w^{-1}(T_w^{(j)}) \times U^v$) and $(T_w^{(n)})^{-1}$ for $j = 1, ..., n$ and for w a place of F which is split over a place $v \not\in T$ of F^+. (Note that $T_{w^c}^{(j)} = (T_w^{(n)})^{-1} T_w^{(n-j)}$, so we need only consider one place w above a given place v of F^+.) If X is a $\mathbf{T}^T_{a,\{\rho_v\}}(U)$-stable subspace of $S_{a,\{\rho_v\},\{\chi_v\}}(U,K)$ then we will write

$$\mathbf{T}^T(X)$$

for the image of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)$ in $\mathrm{End}\,_K(X)$.

Note that $\mathbf{T}^T(X)$ is finite and free as a $\mathcal{O}$-module. Also by corollary 3.3.3 we see that it is reduced.

If $v \in Q$ and $\alpha \in F_{\widetilde{v}}^\times$ write

$$V_\alpha = i_{\widetilde{v}}^{-1} \left(U_1(\widetilde{v}) \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix} U_1(\widetilde{v}) \right) \times U^v.$$

Lemma 3.4.1 *Suppose that for all $v \in R$ the $\mathcal{O}^\times$-valued characters χ_v and χ'_v of $\mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v})$ are congruent modulo λ. Set $V = U^R \times \prod_{v \in R}(i_{\widetilde{v}}^{-1}\mathrm{Iw}_1(\widetilde{v}))$. Then*

$$S_{a,\{\rho_v\},\{\chi_v\}}(U,k) = S_{a,\{\rho_v\},\{\chi'_v\}}(U,k)$$

as $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(V)$-modules. In particular if $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(V)$, then

$$S_{a,\{\rho_v\},\{\chi_v\}}(U,K)_{\mathfrak{m}} \neq (0)$$

if and only if

$$S_{a,\{\rho_v\},\{\chi'_v\}}(U,K)_{\mathfrak{m}} \neq (0).$$

Proof: The first part is immediate from the definitions. The second part follows because $S_{a,\{\rho_v\},\{\chi_v\}}(U,K)_{\mathfrak{m}} \neq (0)$ if and only if $S_{a,\{\rho_v\},\{\chi_v\}}(U,k)_{\mathfrak{m}} \neq (0)$. (The second step uses the assumption that U is sufficiently small so that $S_{a,\{\rho_v\},\{\chi_v\}}(U,\mathcal{O})_{\mathfrak{m}}$ is $\mathcal{O}$-torsion free and

$$S_{a,\{\rho_v\},\{\chi_v\}}(U,K)_{\mathfrak{m}} = S_{a,\{\rho_v\},\{\chi_v\}}(U,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} K$$

and

$$S_{a,\{\rho_v\},\{\chi_v\}}(U,k)_{\mathfrak{m}} = S_{a,\{\rho_v\},\{\chi_v\}}(U,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} k.)$$

□

Proposition 3.4.2 *Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)$. Then there is a unique continuous semisimple representation*

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m})$$

with the following properties. The first two of these properties already characterise $\overline{r}_{\mathfrak{m}}$ uniquely.

1. *$\overline{r}_{\mathfrak{m}}$ is unramified at all but finitely many places.*
2. *If a place $v \notin T$ of F^+ splits as ww^c in F then $\overline{r}_{\mathfrak{m}}$ is unramified at w and $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_w)$ has characteristic polynomial*

$$X^n - T_w^{(1)}X^{n-1} + \dots + (-1)^j(\mathbf{N}w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + \\ + \dots + (-1)^n(\mathbf{N}w)^{n(n-1)/2}T_w^{(n)}.$$

3. *If a place v of F^+ is inert and unramified in F and if U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$, then $\overline{r}_{\mathfrak{m}}$ is unramified above v.*
4. *$\overline{r}^c_{\mathfrak{m}} \cong \overline{r}^\vee_{\mathfrak{m}} \otimes \epsilon^{1-n}$.*
5. *If $v \in S(B)$ and $U_v = G(\mathcal{O}_{F_v^+})$ then $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ is $\widetilde{r}_{\widetilde{v}}$-discrete series. (See definition 2.4.24.)*
6. *Suppose that $w \in \widetilde{S}_l$ is unramified over l, that $U_{w|_{F^+}} = G(\mathcal{O}_{F^+,w})$ and that for each $\tau \in \widetilde{I}_l$ above w we have*

$$l - 1 - n \geq a_{\tau,1} \geq \dots \geq a_{\tau,n} \geq 0.$$

Then

$$\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_w/F_w)} = \mathbf{G}_w(\overline{M}_{\mathfrak{m},w})$$

for some object $\overline{M}_{\mathfrak{m},w}$ of $\mathcal{MF}_{\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m},w}$. Moreover for all $\tau \in \widetilde{I}_l$ over w we have

$$\dim_{\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m}}(\mathrm{gr}^i\overline{M}_{\mathfrak{m},w}) \otimes_{\tau\otimes 1} \mathcal{O} = 1$$

if $i = a_{\tau,j} + n - j$ for some $j = 1, \dots, n$ and $= 0$ otherwise.

Proof: Choose a minimal prime ideal $\wp \subset \mathfrak{m}$ and an irreducible constituent π of $S_{a,\{\rho_v\},\{\chi_v\}}(\{1\},\overline{K})$ such that $\pi^U \neq (0)$ and $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)$ acts on π^U via the quotient $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\wp$. Choosing an invariant lattice in r_π, reducing and semisimplifying gives us the desired representation $\overline{r}_{\mathfrak{m}}$, except that it is defined over the algebraic closure of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m}$. However, as the characteristic polynomial of every element of the image of $\overline{r}_{\mathfrak{m}}$ is rational over $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m}$ and as $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m}$ is a finite field we see that (after conjugation) we may assume that

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\mathfrak{m}).$$

□

Definition 3.4.3 *We will call* $\mathfrak{m}$ Eisenstein *if* $\overline{r}_{\mathfrak{m}}$ *is absolutely reducible.*

Proposition 3.4.4 *Suppose that* $\mathfrak{m}$ *is a non-Eisenstein maximal ideal of the Hecke algebra* $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)$ *with residue field* k. *Then* $\overline{r}_{\mathfrak{m}}$ *has an extension to a continuous homomorphism*

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k).$$

Pick such an extension. There is a unique continuous lifting

$$r_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}})$$

of $\overline{r}_{\mathfrak{m}}$ *with the following properties. The first two of these properties already characterise the lifting* $r_{\mathfrak{m}}$ *uniquely.*

1. $r_{\mathfrak{m}}$ *is unramified at all but finitely many places.*
2. *If a place* $v \notin T$ *of* F^+ *splits as* ww^c *in* F *then* $r_{\mathfrak{m}}$ *is unramified at* w *and* $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ *has characteristic polynomial*

$$X^n - T_w^{(1)}X^{n-1} + \ldots + (-1)^j(\mathbf{N}w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + \ldots + (-1)^n(\mathbf{N}w)^{n(n-1)/2}T_w^{(n)}.$$

3. *If a place* v *of* F^+ *such that* v *is inert and unramified in* F *and if* U_v *is a hyperspecial maximal compact subgroup of* $G(F_v^+)$ *then* $r_{\mathfrak{m}}$ *is unramified at* v.
4. $\nu \circ r_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$, *where* δ_{F/F^+} *denotes the nontrivial character of* $\mathrm{Gal}\,(F/F^+)$ *and where* $\mu_{\mathfrak{m}} \in \mathbf{Z}/2\mathbf{Z}$.
5. *Suppose that* $w \in \widetilde{S}_l$ *is unramified over* l, *that* $U_{w|_{F^+}} = G(\mathcal{O}_{F^+,w})$ *and that for each* $\tau \in \widetilde{I}_l$ *above* w *we have*

$$l - 1 - n \geq a_{\tau,1} \geq \ldots \geq a_{\tau,n} \geq 0.$$

Then for each open ideal $I \subset \mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}}$

$$(r_{\mathfrak{m}} \otimes_{\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}}} \mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}}/I)|_{\mathrm{Gal}\,(\overline{F}_w/F_w)} = \mathbf{G}_w(M_{\mathfrak{m},I,w})$$

for some object $M_{\mathfrak{m},I,w}$ *of* $\mathcal{MF}_{\mathcal{O},w}$.

6. If $v \in S(B)$ and $U_v = G(\mathcal{O}_{F^+,v})$ then $r_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ is $\widetilde{r}_{\widetilde{v}}$-discrete series. (See definition 2.4.24.)

7. If $v \in R$ and $\sigma \in I_{F_{\widetilde{v}}}$ then $r_{\mathfrak{m}}(\sigma)$ has characteristic polynomial

$$\prod_{j=1}^{n}(X - \chi_{v,j}^{-1}(\mathrm{Art}\,_{F_{\widetilde{v}}}^{-1}\sigma)).$$

8. Suppose that $v \in Q$. Let $\phi_{\widetilde{v}}$ be a lift of $\mathrm{Frob}_{\widetilde{v}}$ to $\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ and let $\varpi_{\widetilde{v}}$ be an element of $F_{\widetilde{v}}^{\times}$ such that $\mathrm{Art}\,_{F_{\widetilde{v}}}\varpi_{\widetilde{v}} = \phi_{\widetilde{v}}$ on the maximal abelien extension of $F_{\widetilde{v}}$. Suppose that $\alpha \in k$ is a simple root of the characteristic polynomial of $\overline{r}_{\mathfrak{m}}(\phi_{\widetilde{v}})$. Then there is a unique root $\widetilde{\alpha} \in \mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}}$ of the characteristic polynomial of $r_{\mathfrak{m}}(\phi_{\widetilde{v}})$ which lifts α.
Suppose further that Y is a $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)[V_{\varpi_{\widetilde{v}}}]$-invariant subspace of the space $S_{a,\{\rho_v\},\{\chi_v\}}(U,K)_{\mathfrak{m}}$ such that $V_{\varpi_{\widetilde{v}}} - \alpha$ is topologically nilpotent on Y. Then for each $\beta \in F_{\widetilde{v}}^{\times}$ with non-negative valuation the element V_β (in $\mathrm{End}\,_K(Y)$) lies in $\mathbf{T}^T(Y)$. Moreover $\beta \mapsto V_\beta$ extends to a continuous character $V : F_{\widetilde{v}}^{\times} \to \mathbf{T}^T(Y)^{\times}$. Further $(X - V_{\varpi_{\widetilde{v}}})$ divides the characteristic polynomial of $r_{\mathfrak{m}}(\phi_{\widetilde{v}})$ over $\mathbf{T}^T(Y)$.
If $\mathbf{N}v \equiv 1 \bmod l$ then

$$r_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})} = s \oplus (V \circ \mathrm{Art}\,_{F_{\widetilde{v}}}^{-1}),$$

where s is unramified.

Proof: By lemma 2.1.4 we can extend $\overline{r}_{\mathfrak{m}}$ to a homomorphism

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k)$$

with $\nu \circ \overline{r}_{\mathfrak{m}} = \epsilon^{n-1}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$ and $\overline{r}_{\mathfrak{m}}(c_v) \notin GL_n(k)$ for any infinite place v of F^+. Moreover, up to $GL_n(k)$-conjugation, the choices of such extensions are parametrised by $k^\times/(k^\times)^2$.

Similarly, for any minimal primes $\wp \subset \mathfrak{m}$ we have a continuous homomorphism $r_\wp$ from $\mathrm{Gal}\,(\overline{F}/F^+)$ to the points of $\mathcal{G}_n$ over the algebraic closure of $\mathbf{Q}_l$ in the algebraic closure of the field of fractions of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\wp$ such that

– $r_\wp$ is unramified almost everywhere;
– $r_\wp^{-1}GL_n = \mathrm{Gal}\,(\overline{F}/F)$; and
– for all places $v \notin T$ of F^+ which split $v = ww^c$ in F the characteristic polynomial of $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ is

$$X^n - T_w^{(1)}X^{n-1} + \ldots + (-1)^j(\mathbf{N}w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + \ldots + (-1)^n(\mathbf{N}w)^{n(n-1)/2}T_w^{(n)}.$$

According to lemma 2.1.5 we may assume that $r_\wp$ is actually valued in $\mathcal{G}_n(\mathcal{O}_\wp)$ where $\mathcal{O}_\wp$ is the ring of integers of some finite extension of the field of fractions of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)/\wp$. Then by lemma 2.1.4 again we may assume that the reduction of $r_\wp$ modulo the maximal ideal of $\mathcal{O}_\wp$ *equals* $\overline{r}_\mathfrak{m}$. (Not simply conjugate to $\overline{r}_\mathfrak{m}$.) Let A denote the subring of $k \oplus \bigoplus_{\wp\subset\mathfrak{m}} \mathcal{O}_\wp$ consisting of elements $(a_\mathfrak{m}, a_\wp)$ such that for all $\wp$ the reduction of $a_\wp$ modulo the maximal ideal of $\mathcal{O}_\wp$ is $a_\mathfrak{m}$. Then

$$\overline{r}_\mathfrak{m} \oplus \bigoplus_\wp r_\wp : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(A).$$

Moreover the natural map

$$\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_\mathfrak{m} \longrightarrow A$$

is an injection. (Because $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_\mathfrak{m}$ is reduced.) Thus by lemma 2.1.12 we see that $\overline{r}_\mathfrak{m} \oplus \bigoplus_\wp r_\wp$ is $GL_n(A)$ conjugate to a representation

$$r_\mathfrak{m} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_\mathfrak{m})$$

such that:

– If a place $v \notin T$ of F^+ splits as ww^c in F then $r_\mathfrak{m}$ is unramified at w and $r_\mathfrak{m}(\mathrm{Frob}_w)$ has characteristic polynomial

$$X^n - T_w^{(1)} X^{n-1} + \ldots + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \ldots + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T_w^{(n)}.$$

– If a place v of F^+ is inert and unramified in F and if U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$ then $r_\mathfrak{m}$ is unramified at v.

It is easy to verify that $r_\mathfrak{m}$ also satisfies properties 4 and 5 of the proposition.

We next turn to part 6. After base changing to an algebraically closed field each $r_\wp|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ has a unique filtration such that $\mathrm{gr}^0 r_\wp|_{I_{F_{\widetilde{v}}}} \cong \tilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}}$, and

$$\mathrm{gr}^i r_\wp|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})} \cong (\mathrm{gr}^0 r_\wp|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})})(\epsilon^i)$$

for $i = 0, \ldots, m_v - 1$ (and $= (0)$ otherwise). Enlarging $\mathcal{O}_\wp$ if need be we may assume that this filtration is defined over the field of fractions of $\mathcal{O}_\wp$. As $\tilde{r}_{\widetilde{v}} \otimes_\mathcal{O} k$ is irreducible, such a filtration also exists over $\mathcal{O}_\wp$. Because of the uniqueness of the filtration $\overline{\mathrm{Fil}}^i$ on the base change of $\overline{r}_\mathfrak{m}$ to the residue field of $\mathcal{O}_\wp$ we see that these filtrations piece together to give a filtration of $\overline{r}_\mathfrak{m} \oplus \bigoplus_\wp r_\wp$ over A. As the isomorphisms $\overline{\mathrm{gr}}^i \overline{r}_\mathfrak{m} \cong (\overline{\mathrm{gr}}^0 \overline{r}_\mathfrak{m})(\epsilon^i)$ are unique up to scalar multiples we get isomorphisms

$$\mathrm{gr}^i(\overline{r}_\mathfrak{m} \oplus \bigoplus_\wp r_\wp) \cong (\mathrm{gr}^0(\overline{r}_\mathfrak{m} \oplus \bigoplus_\wp r_\wp))(\epsilon^i)$$

over $A[\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})]$ which are compatible with the chosen isomorphism between $\overline{\mathrm{gr}}^i\overline{r}_{\mathfrak{m}}$ and $(\overline{\mathrm{gr}}^0\overline{r}_{\mathfrak{m}})(\epsilon^i)$. As

$$Z_{GL_{n/m_v}(\mathcal{O}_\wp)}(\mathrm{gr}\,^0 r_\wp(I_{F_{\widetilde{v}}})) \twoheadrightarrow Z_{GL_{n/m_v}(\mathcal{O}_\wp/\mathfrak{m}_{\mathcal{O}_\wp})}(\mathrm{gr}\,^0\overline{r}_{\mathfrak{m}}(I_{F_{\widetilde{v}}}))$$

(see lemma 2.4.23), we see that we get an isomorphism

$$\mathrm{gr}\,^0(\overline{r}_{\mathfrak{m}} \oplus \bigoplus_\wp r_\wp) \cong \widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} A$$

over $A[I_{F_w}]$ compatible with the chosen isomorphism $\overline{\mathrm{gr}}^0\overline{r}_{\mathfrak{m}} \cong \widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k$. Thus $\overline{r}_{\mathfrak{m}} \oplus \bigoplus_\wp r_\wp$ is $\widetilde{r}_{\widetilde{v}}$-discrete series. It follows that $r_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ is also $\widetilde{r}_{\widetilde{v}}$-discrete series.

Part 7 follows from proposition 3.3.4 and lemma 3.1.6. (Note that the space $S_{a,\{\rho_v\},\{\chi_v\}}(U,K)$ equals the subspace of $S_{a,\{\rho_v\},\emptyset}(U^R \times \prod_{v\in R} i_{\widetilde{v}}^{-1}\mathrm{Iw}_1(\widetilde{v}),K)$ on which $\mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v})$ acts by χ_v^{-1}.)

Finally we turn to part 8 of the proposition. The existence of $\widetilde{\alpha}$ follows at once from Hensel's lemma. Let $P(X) \in \mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}}[X]$ denote the characteristic polynomial of $r_{\mathfrak{m}}(\phi_{\widetilde{v}})$. Thus $P(X) = (X-\widetilde{\alpha})Q(X)$ where $Q(\widetilde{\alpha}) \in \mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)^\times_{\mathfrak{m}}$.

Write $Y \otimes_K \overline{K} = \bigoplus((Y \otimes \overline{K}) \cap \pi)$ as π runs over irreducible smooth representations of $G(\mathbf{A}^\infty_{F^+})$. From lemmas 3.1.3 and 3.1.5 and the fact that $V_{\varpi_{\widetilde{v}}} - \alpha$ is topologically nilpotent we see that $\dim((Y\otimes\overline{K})\cap\pi) \leq 1$ for all π. Let $\phi'_{\widetilde{v}}$ be any lift of $\mathrm{Frob}_{\widetilde{v}}$ to $\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ and let $\mathrm{Art}_{F_{\widetilde{v}}}\varpi'_{\widetilde{v}} = \phi'_{\widetilde{v}}$. Let P' denote the characteristic polynomial of $r_{\mathfrak{m}}(\phi'_{\widetilde{v}})$ and let $\widetilde{\alpha}'$ be its unique root in $\mathbf{T}^T(Y)$ over α. As $V_{\varpi_{\widetilde{v}}}$ and $V_{\varpi'_{\widetilde{v}}}$ commute, each $(Y\otimes\overline{K})\cap\pi$ is invariant under $V_{\varpi'_{\widetilde{v}}}$. By lemma 3.1.5 $V_{\varpi'_{\widetilde{v}}}V^{-1}_{\varpi_{\widetilde{v}}}$ is topologically unipotent on $(Y\otimes\overline{K})\cap\pi$. Lemmas 3.1.3 and 3.1.5 imply that $P'(V_{\varpi'_{\widetilde{v}}}) = 0$ on $(Y\otimes\overline{K})\cap\pi$. Thus $V_{\varpi'_{\widetilde{v}}} = \widetilde{\alpha}'$ on $(Y\otimes\overline{K})\cap\pi$. Hence $V_{\varpi'_{\widetilde{v}}} = \widetilde{\alpha}' \in \mathbf{T}^T(Y) \subset \mathrm{End}\,_K(Y)$. It follows that $V_\beta \in \mathbf{T}^T(Y)$ for all $\beta \in F^\times_{\widetilde{v}}$ with non-negative valuation and that $\beta \mapsto V_\beta$ extends to a continuous character $V : F^\times_{\widetilde{v}} \to \mathbf{T}^T(Y)^\times$.

Now suppose that $\mathbf{N}v \equiv 1 \bmod l$. From lemma 3.1.5 we see that if $(Y\otimes\overline{K})\cap\pi \neq (0)$ then either π is unramified or $\pi^{U_0(\widetilde{v})} = (0)$ (otherwise $V_{\varpi_{\widetilde{v}}}$ would be a multiple root of the characteristic polynomial of $\overline{r}_{\mathfrak{m}}(\phi_{\widetilde{v}})$). Thus $(r_{\mathfrak{m}} \otimes \mathbf{T}^T(Y))(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}))$ is abelian. We have a decomposition

$$\mathbf{T}^T(Y)^n = Q(\phi_{\widetilde{v}})\mathbf{T}^T(Y)^n \oplus (\phi_{\widetilde{v}} - \widetilde{\alpha})\mathbf{T}^T(Y)^n.$$

As $(r_{\mathfrak{m}} \otimes \mathbf{T}^T(Y))(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}))$ is abelian we see that this decomposition is preserved by $\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$. By lemma 3.1.5 we see that after projection to any $\pi\cap(Y\otimes\overline{K})$, $\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ acts on $Q(\phi_{\widetilde{v}})\mathbf{T}^T(Y)^n$ by $V_\pi \circ \mathrm{Art}^{-1}_{F_w\widetilde{v}}$ and its action on $(\phi_{\widetilde{v}} - \widetilde{\alpha})\mathbf{T}^T(Y)^n$ is unramified. We conclude that $\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ acts on $Q(\phi_{\widetilde{v}})\mathbf{T}^T(Y)^n$ by V and that its action on $(\phi_{\widetilde{v}}-\widetilde{\alpha})\mathbf{T}^T(Y)^n$ is unramified. This completes the proof of part 8 of the proposition. $\square$

Corollary 3.4.5 *Suppose that $\mathfrak{m}$ is a non-Eisenstein maximal ideal of the Hecke algebra $\mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)$. Suppose also that $v \in T-(S(B)\cup S_l)$ and that $U_v = G(\mathcal{O}_{F^+,v})$. If w is a prime of F above v then for $j=1,...,n$ we have*

$$T_w^{(j)} \in \mathbf{T}^T_{a,\{\rho_v\},\{\chi_v\}}(U)_{\mathfrak{m}} \subset \mathrm{End}\,(S_{a,\{\rho_v\},\{\chi_v\}}(U,K)_{\mathfrak{m}}).$$

Proof: One need only remark that

$$T_w^{(j)} = (\mathbf{N}w)^{j(1-j)/2}\mathrm{tr}\, \wedge^j r_{\mathfrak{m}}(\mathrm{Frob}_w).$$

$\square$

3.5. *$R=\mathbf{T}$ theorems: the minimal case.* — In this section we will prove the quality of certain global Galois deformation rings and certain Hecke algebras in the so called 'minimal case'. The results of this section are not required for the proofs of the main theorems in [Tay] and [HSBT]. It could be skipped by those only interested in these applications, but it might serve as a good warm up for understanding the arguments of [Tay].

We must first establish some notation and assumptions. In the interests of clarity we recapitulate all running assumptions made in previous sections.

Fix a positive integer $n \geq 2$ and a prime $l > n$.

Fix an imaginary quadratic field E in which l splits and a totally real field F^+ such that

- $F = F^+E/F^+$ is unramified at all finite primes, and
- $F^+/\mathbf{Q}$ is unramified at l.

Fix a finite non-empty set of places $S(B)$ of places of F^+ with the following properties:

- Every element of $S(B)$ splits in F.
- $S(B)$ contains no place above l.
- If n is even then

$$n[F^+:\mathbf{Q}]/2 + \#S(B) \equiv 0 \bmod 2.$$

Choose a division algebra B with centre F with the following properties:

- $\dim_F B = n^2$.
- $B^{\mathrm{op}} \cong B \otimes_{E,c} E$.
- B splits outside $S(B)$.

– If w is a prime of F above an element of $S(B)$, then B_w is a division algebra.

Fix an involution $\ddagger$ on B and define an algebraic group G/F^+ by

$$G(A) = \{g \in B \otimes_{F^+} A : \; g^{\ddagger \otimes 1} g = 1\},$$

such that

– $\ddagger|_F = c$,
– for a place $v|\infty$ of F^+ we have $G(F_v^+) \cong U(n)$, and
– for a finite place $v \notin S(B)$ of F^+ the group $G(F_v^+)$ is quasi-split.

The purpose of the assumption that $S(B) \neq \emptyset$ is to simplify the use of the trace formula in relating automorphic forms on G to automorphic forms on GL_n/F and in attaching Galois representations to automorphic forms on G.

Choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^{\ddagger} = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is maximal for all primes w of F which are split over F^+. This gives a model of G over $\mathcal{O}_{F^+}$. If $v \notin S(B)$ is a prime of F^+ which splits in F choose an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F,v})$ such that $i_v(x^{\ddagger}) = {}^t i_v(x)^c$. If w is a prime of F above v this gives rise to an isomorphism $i_w : G(F_v^+) \xrightarrow{\sim} GL_n(F_w)$ as in section 3.3. If $v \in S(B)$ and w is a prime of F above v choose isomorphisms $i_w : G(F_v^+) \xrightarrow{\sim} B_w^{\times}$ such that $i_{w^c} = i_w^{-\ddagger}$ and $i_w G(\mathcal{O}_{F^+,v}) = \mathcal{O}_{B,w}^{\times}$.

Let S_l denote the set of primes of F^+ above l. Let S_a denote a non-empty set, disjoint from $S_l \cup S(B)$, of primes of F^+ such that

– if $v \in S_a$ then v splits in F, and
– if $v \in S_a$ lies above a rational prime p then $[F(\zeta_p) : F] > n$.

Let $T = S(B) \cup S_l \cup S_a$. Let $\widetilde{T}$ denote a set of primes of F above T such that $\widetilde{T} \coprod \widetilde{T}^c$ is the set of all primes of F above T. If $v \in T$ we will let $\widetilde{v}$ denote the prime of $\widetilde{T}$ above v. If $S \subset T$ we will let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$.

Let $U = \prod_v U_v$ denote an open compact subgroup of $G(\mathbf{A}_{F^+}^{\infty})$ such that

– if v is not split in F then U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$,
– if $v \notin S_a$ splits in F then $U_v = G(\mathcal{O}_{F^+,v})$,
– if $v \in S_a$ then $U_v = i_{\widetilde{v}}^{-1} \ker(GL_n(\mathcal{O}_{F,\widetilde{v}}) \to GL_n(\mathcal{O}_{F,\widetilde{v}}/(\varpi_{\widetilde{v}})))$.

Then U is sufficiently small. (The purpose of the non-empty set S_a is to ensure this.)

Let $K/\mathbf{Q}_l$ be a finite extension which contains the image of every embedding $F^+ \hookrightarrow \overline{K}$. Let $\mathcal{O}$ denote its ring of integers, λ the maximal ideal of $\mathcal{O}$ and k the residue field $\mathcal{O}/\lambda$.

For each $\tau : F \hookrightarrow K$ choose integers $a_{\tau,1}, ..., a_{\tau,n}$ such that

– $a_{\tau c,i} = -a_{\tau,n+1-i}$, and
– if τ gives rise to a place in $\widetilde{S}_l$ then

$$l-1-n \geq a_{\tau,1} \geq ... \geq a_{\tau,n} \geq 0.$$

For each $v \in S(B)$ let $\rho_v : G(F_v^+) \longrightarrow GL(M_{\rho_v})$ denote a representation of $G(F_v^+)$ on a finite free $\mathcal{O}$-module such that ρ_v has open kernel and $M_{\rho_v} \otimes_{\mathcal{O}} \overline{K}$ is irreducible. For $v \in S(B)$, define m_v, $\pi_{\widetilde{v}}$ and $\widetilde{r}_{\widetilde{v}}$ by

$$\mathrm{JL}\,(\rho_v \circ i_{\widetilde{v}}^{-1}) = \mathrm{Sp}_{\,m_v}(\pi_{\widetilde{v}})$$

and

$$\widetilde{r}_{\widetilde{v}} = r_l(\pi_{\widetilde{v}}|\ \ |^{(n/m_{\widetilde{v}}-1)(1-m_{\widetilde{v}})/2}).$$

We will suppose that

$$\widetilde{r}_{\widetilde{v}} : \mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}) \longrightarrow GL_{n/m_{\widetilde{v}}}(\mathcal{O})$$

(as opposed to $GL_{n/m_{\widetilde{v}}}(\overline{K})$), that the reduction of $\widetilde{r}_{\widetilde{v}} \bmod \lambda$ is absolutely irreducible, that every irreducible subquotient of $\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \bmod \lambda$ is absolutely irreducible, and that for $i = 1, ..., m_v$ we have

$$\widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k \not\cong \widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k(\epsilon^i).$$

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(U)$ with residue field k and let

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k)$$

be a continuous homomorphism associated to $\mathfrak{m}$ as in propositions 3.4.2 and 3.4.4. Note that

$$\nu \circ \overline{r}_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$$

where δ_{F/F^+} is the non-trivial character of $\mathrm{Gal}\,(F/F^+)$ and where $\mu_{\mathfrak{m}} \in \mathbf{Z}/2\mathbf{Z}$. We will *assume* that $\overline{r}_{\mathfrak{m}}$ has the following properties.

– $\overline{r}_{\mathfrak{m}}(\mathrm{Gal}\,(\overline{F}/F^+(\zeta_l)))$ is big in the sense of section 2.5.
– If $v \in S_a$ then $\overline{r}_{\mathfrak{m}}$ is unramified at v and

$$H^0(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), (\mathrm{ad}\,\overline{r}_{\mathfrak{m}})(1)) = (0).$$

We will write $\mathbf{T}_{\mathfrak{m}}$ for the localisation $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(U)_{\mathfrak{m}}$ and $X_{\mathfrak{m}}$ for the localisation $S_{a,\{\rho_v\},\emptyset}(U,\mathcal{O})_{\mathfrak{m}}$. Thus $\mathbf{T}_{\mathfrak{m}}$ is a local, commutative subalgebra of $\mathrm{End}_{\,\mathcal{O}}(X_{\mathfrak{m}})$. It is reduced and finite, free as an $\mathcal{O}$-module. Let

$$r_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbf{T}_{\mathfrak{m}})$$

denote the continuous lifting of $\overline{r}_{\mathfrak{m}}$ provided by proposition 3.4.4. Then $\mathbf{T}_{\mathfrak{m}}$ is generated as an $\mathcal{O}$-algebra by the coefficients of the characteristic polynomials of $r_{\mathfrak{m}}(\sigma)$ for $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$.

Consider the deformation problem $\mathcal{S}$ given by

$$(F/F^+, T, \widetilde{T}, \mathcal{O}, \overline{r}_{\mathfrak{m}}, \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}, \{\mathcal{D}_v\}_{v\in T})$$

where:

– For $v \in S_a$, $\mathcal{D}_v$ will consist of all lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and so

$$L_v = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), \mathrm{ad}\,\overline{r}_{\mathfrak{m}}) = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

– For $v \in S_l$, $\mathcal{D}_v$ and L_v are as described in section 2.4.1 (i.e. consists of crystalline deformations).

– For $v \in S(B)$, $\mathcal{D}_v$ consists of lifts which are $\widetilde{r}_{\widetilde{v}}$-discrete series as described in section 2.4.5.

Also let

$$r_{\mathcal{S}}^{\mathrm{univ}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(R_{\mathcal{S}}^{\mathrm{univ}})$$

denote the universal deformation of $\overline{r}_{\mathfrak{m}}$ of type $\mathcal{S}$. By proposition 3.4.4 there is a natural surjection

$$R_{\mathcal{S}}^{\mathrm{univ}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$$

such that $r_{\mathcal{S}}^{\mathrm{univ}}$ pushes forward to $r_{\mathfrak{m}}$.

We can now state and prove our main result.

Theorem 3.5.1 *Keep the notation and assumptions of the start of this section. Then*

$$R_{\mathcal{S}}^{\mathrm{univ}} \xrightarrow{\sim} \mathbf{T}_{\mathfrak{m}}$$

is an isomorphism of complete intersections and $X_{\mathfrak{m}}$ is free over $\mathbf{T}_{\mathfrak{m}}$. Moreover $\mu_{\mathfrak{m}} \equiv n \bmod 2$.

Proof: To prove this we will appeal to Diamond's and Fujiwara's improvement to Faltings' understanding of the method of [TW]. More precisely we will appeal to theorem 2.1 of [Dia]. We remark that one may easily weaken the hypotheses of this theorem in the following minor ways. The theorem with the weaker hypotheses is easily deduced from the theorem as it is stated in [Dia]. In the notation of [Dia] one can take $B = k[[X_1, ..., X_{r'}]]$ with $r' \leq r$. Also in place of his assumption (c) one need only assume that H_n is free over $A/\mathfrak{n}_n$, where $\{\mathfrak{n}_n\}$ is a family of open ideals contained in $\mathfrak{n}$ with the property that $\bigcap_n \mathfrak{n}_n = (0)$. We also remark with these weakened hypotheses one may also deduce from the proof of theorem 2.1 of [Dia] that in fact $r = r'$.

Choose an integer q as in proposition 2.5.9. Set

$$q' = q - n[F^+ : \mathbf{Q}](1 + (-1)^{n-1+\mu_{\mathfrak{m}}})/2.$$

For each $N \in \mathbf{Z}_{\geq 1}$ choose $(Q_N, \widetilde{Q}_N, \{\overline{\psi}_v^{(N)}\}_{v \in Q_N})$ as in proposition 2.5.9 and definition 2.5.7. We will use the notations $\mathcal{S}(Q_N)$, Δ_v, Δ_{Q_N} and $\mathfrak{a}_{\emptyset, Q_N}$ as in definition 2.5.7. Recall that

$$(R^{\mathrm{univ}}_{\mathcal{S}(Q_N)})_{\Delta_{Q_N}} = R^{\mathrm{univ}}_{\mathcal{S}}.$$

By proposition 2.5.9 there is a surjection

$$\mathcal{O}[[Y_1, ..., Y_{q'}]] \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}(Q_N)}.$$

Let ψ_N denote the composite

$$\psi_N : \mathcal{O}[[Y_1, ..., Y_{q'}]] \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}(Q_N)} \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}}.$$

There is a surjection

$$\mathcal{O}[[Z_1, ..., Z_q]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}]$$

such that, if $\mathfrak{n}_N$ denotes the kernel, then $\bigcap_N \mathfrak{n}_N = (0)$. We can lift the map

$$\mathcal{O}[[Z_1, ..., Z_q]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}] \longrightarrow R^{\mathrm{univ}}_{\mathcal{S}(Q_N)}$$

to a map

$$\phi_N : \mathcal{O}[[Z_1, ..., Z_q]] \longrightarrow \mathcal{O}[[Y_1, ..., Y_{q'}]].$$

Then the composite

$$\mathcal{O}[[Z_1, ..., Z_q]] \stackrel{\psi_N \circ \phi_N}{\longrightarrow} R^{\mathrm{univ}}_{\mathcal{S}}/\lambda$$

has kernel $(\lambda, Z_1, ..., Z_q)$.

Note that $X_{\mathfrak{m}}$ is a $R^{\mathrm{univ}}_{\mathcal{S}}$-module via $R^{\mathrm{univ}}_{\mathcal{S}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$.

Define open compact subgroups $U_1(Q_N) = \prod_v U_1(Q_N)_v$ and $U_0(Q_N) = \prod_v U_0(Q_N)_v$ of $G(\mathbf{A}^{\infty}_{F^+})$ by

- $U_1(Q_N)_v = U_0(Q_N)_v = U_v$ if $v \notin Q_N$,
- $U_1(Q_N)_v = i_{\widetilde{v}}^{-1} U_1(\widetilde{v})$ if $v \in Q_N$, and
- $U_0(Q_N)_v = i_{\widetilde{v}}^{-1} U_0(\widetilde{v})$ if $v \in Q_N$.

By corollary 3.4.5 we see that we have

$$\mathbf{T}^{T \cup Q_N}_{a, \{\rho_v\}, \emptyset}(U_1(Q_N))_{\mathfrak{m}} \twoheadrightarrow \mathbf{T}^{T \cup Q_N}_{a, \{\rho_v\}, \emptyset}(U_0(Q_N))_{\mathfrak{m}} \twoheadrightarrow \mathbf{T}^{T \cup Q_N}_{a, \{\rho_v\}, \emptyset}(U)_{\mathfrak{m}} = \mathbf{T}^{T}_{a, \{\rho_v\}, \emptyset}(U)_{\mathfrak{m}}.$$

For $v \in Q_N$ choose $\phi_{\widetilde{v}} \in \mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ lifting $\mathrm{Frob}_{\widetilde{v}}$ and $\varpi_{\widetilde{v}} \in F_{\widetilde{v}}^{\times}$ with $\phi_{\widetilde{v}} = \mathrm{Art}_{F_{\widetilde{v}}} \varpi_{\widetilde{v}}$ on the maximal abelian extension of $F_{\widetilde{v}}$. Let

$$P_{\widetilde{v}} \in \mathbf{T}^{T \cup Q_N}_{a, \{\rho_v\}, \emptyset}(U_1(Q_N))_{\mathfrak{m}}[X]$$

denote the characteristic polynomial of $r_{\mathfrak{m}}(\phi_{\tilde{v}})$. By Hensel's lemma we have a unique factorisation

$$P_{\tilde{v}}(X) = (X - A_{\tilde{v}})Q_{\tilde{v}}(X)$$

over $\mathbf{T}^{T\cup Q_N}_{a,\{\rho_v\},\emptyset}(U_1(Q_N))_{\mathfrak{m}}$, where $A_{\tilde{v}}$ lifts $\overline{\psi}_v^{(N)}(\mathrm{Frob}_{\tilde{v}})$ and where $Q_{\tilde{v}}(A_{\tilde{v}})$ is a unit in $\mathbf{T}^{T\cup Q_N}_{a,\{\rho_v\},\emptyset}(U_1(Q_N))_{\mathfrak{m}}$. By lemmas 3.1.3 and 3.1.5 we see that $P_{\tilde{v}}(V_{\varpi_{\tilde{v}}}) = 0$ on $S_{a,\{\rho_v\},\emptyset}(U_1(Q_N),\mathcal{O})_{\mathfrak{m}}$. Set

$$H_{1,Q_N} = (\prod_{v\in Q_N} Q_{\tilde{v}}(V_{\varpi_{\tilde{v}}}))S_{a,\{\rho_v\},\emptyset}(U_1(Q_N),\mathcal{O})_{\mathfrak{m}}$$

and

$$H_{0,Q_N} = (\prod_{v\in Q_N} Q_{\tilde{v}}(V_{\varpi_{\tilde{v}}}))S_{a,\{\rho_v\},\emptyset}(U_0(Q_N),\mathcal{O})_{\mathfrak{m}}.$$

We see that H_{1,Q_N} is a $\mathbf{T}^{T\cup Q_N}_{a,\{\rho_v\},\emptyset}(U_1(Q_N))$-direct summand of the larger module $S_{a,\{\rho_v\},\emptyset}(U_1(Q_N),\mathcal{O})$, and hence by lemma 3.3.1

$$\mathrm{tr}_{\,U_0(Q_N)/U_1(Q_N)} : (H_{1,Q_N})_{U_0(Q_N)/U_1(Q_N)} \xrightarrow{\sim} H_{0,Q_N}.$$

Moreover for all $v \in Q_N$, $V_{\varpi_{\tilde{v}}} = A_{\tilde{v}}$ on H_{1,Q_N}. By part 7 of proposition 3.4.4 we see that for each $v \in Q_N$ there is a character

$$V_{\tilde{v}} : F_{\tilde{v}}^{\times} \longrightarrow \mathbf{T}^{T\cup Q_N}(H_{1,Q_N})^{\times}$$

such that

- if $\alpha \in F_{\tilde{v}}^{\times} \cap \mathcal{O}_{F,\tilde{v}}$ then $V_{\tilde{v}}(\alpha) = V_\alpha$ on H_{1,Q_N}, and
- $r_{\mathfrak{m}}|_{W_{F_{\tilde{v}}}} = s \oplus (V_{\tilde{v}} \circ \mathrm{Art}_{F_{\tilde{v}}}^{-1})$ where s is unramified.

Thus $r_{\mathfrak{m}}$ gives rise to a surjection

$$R^{\mathrm{univ}}_{\mathcal{S}(Q_N)} \twoheadrightarrow \mathbf{T}^{T\cup Q_N}(H_{Q_N}).$$

The composite

$$\prod_{v\in Q_N} \mathcal{O}_{F,\tilde{v}}^{\times} \twoheadrightarrow \Delta_{Q_N} \longrightarrow (R^{\mathrm{univ}}_{\mathcal{S}(Q_N)})^{\times} \longrightarrow \mathbf{T}^{T\cup Q_N}(H_{Q_N})^{\times}$$

is just $\prod_v V_{\tilde{v}}$. As H_{1,Q_N} is a direct summand of $S_{a,\{\rho_v\},\emptyset}(U_1(Q_N),\mathcal{O})$ over $\mathbf{T}^{T\cup Q_N}_{a,\{\rho_v\},\emptyset}(U_1(Q_N))$, lemma 3.3.1 now tells us that H_{1,Q_N} is a free $\mathcal{O}[\Delta_{Q_N}]$-module and that

$$(H_{1,Q_N})_{\Delta_{Q_N}} \xrightarrow{\sim} H_{0,Q_N}.$$

Also lemma 3.2.2, combined with lemma 3.1.5, tells us that

$$(\prod_{v\in Q_N} Q_{\tilde{v}}(V_{\varpi_t v})) : X_{\mathfrak{m}} \xrightarrow{\sim} H_{0,Q_N}.$$

Now we apply theorem 2.1 of [Dia] (as reformulated in the first paragraph of this proof) to $A = k[[Z_1, ..., Z_q]]$, $B = k[[Y_1, ..., Y_{q'}]]$, $R = R_{\mathcal{S}}^{\mathrm{univ}}/\lambda$, $H = X_{\mathfrak{m}}/\lambda$ and $H_N = H_{1,Q_N}/\lambda$. We deduce that $r = r'$, that $X_{\mathfrak{m}}/\lambda$ is free over $R_{\mathcal{S}}^{\mathrm{univ}}/\lambda$ via $R_{\mathcal{S}}^{\mathrm{univ}}/\lambda \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}/\lambda$ and that $R_{\mathcal{S}}^{\mathrm{univ}}/\lambda$ is a complete intersection. As $X_{\mathfrak{m}}$ is free over $\mathcal{O}$ we see that $X_{\mathfrak{m}}$ is also free over $R_{\mathcal{S}}^{\mathrm{univ}}$ via $R_{\mathcal{S}}^{\mathrm{univ}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$. Thus $R_{\mathcal{S}}^{\mathrm{univ}} \xrightarrow{\sim} \mathbf{T}_{\mathfrak{m}}$ is free over $\mathcal{O}$ and hence a complete intersection. The equality $q = q'$ tells us that $\mu_{\mathfrak{m}} \equiv n \bmod 2$. $\square$

4. Automorphic forms on GL_n.

In this chapter we will recall some general facts about the relationship between automorphic forms on GL_n and Galois representations. We will then combine theorem 3.5.1 with some instances of base change to obtain modularity lifting theorems for GL_n.

4.1. *Characters..* — The first three lemmas are well known.

Lemma 4.1.1 *Suppose that F is a number field and that S is a finite set of places of F. Suppose also that*

$$\chi_S : \prod_{v \in S} F_v^\times \longrightarrow \overline{\mathbf{Q}}^\times$$

is a continuous character of finite order. Then there is a continuous character

$$\chi : F^\times \backslash \mathbf{A}_F^\times \longrightarrow \overline{\mathbf{Q}}^\times$$

such that $\chi|_{\prod_{v\in S} F_v^\times} = \chi_S$.

Proof: One may suppose that S contains all infinite places. Then we choose an open subgroup $U \subset (\mathbf{A}_F^S)^\times$ such that χ_S is trivial on $U \cap F^\times$. (This is possible as any finite index subgroup of $\mathcal{O}_F^\times$ is a congruence subgroup.) Then we can extend χ_S to $U \prod_{v\in S} F_v^\times/(U \cap F^\times)$ by setting it to one on U. Finally we can extend this character to $\mathbf{A}_F^\times/F^\times$ (which contains $U \prod_{v\in S} F_v^\times/(U \cap F^\times)$ as an open subgroup). $\Box$

Lemma 4.1.2 *Suppose that F is a number field, D/F is a finite Galois extension and S is a finite set of places of F. For $v \in S$ let E_v'/F_v be a finite Galois extension. Then we can find a finite, soluble Galois extension E/F linearly disjoint from D such that for each $v \in S$ and each prime w of E above v, the extension E_w/F_v is isomorphic to E_v'/F_v.*

Proof: For each $D \supset D_i \supset F$ with D_i/F Galois with a simple Galois group, choose a prime $v_i \notin S$ of F which does not split completely in D_i. Add the v_i to S along with $E'_{v_i} = F_{v_i}$. Then we can drop the condition that E/F is disjoint from D/F - it will be automatically satisfied.

Using induction on the maximum of the degrees $[E_v' : F_v]$ we may reduce to the case that each E_v'/F_v is cyclic. Then we can choose a continuous finite order character

$$\chi_S : \prod_{v \in S} F_v^\times \longrightarrow \overline{\mathbf{Q}}^\times$$

such that $\ker \chi_S|_{F_v^\times}$ corresponds (under local class field theory) to E'_v/F_v for all $v \in S$. According to the previous lemma we can extend χ to a continuous character

$$\chi : F^\times \backslash \mathbf{A}_F^\times \longrightarrow \overline{\mathbf{Q}}^\times.$$

Let E/F correspond, under global class field theory, to $\ker \chi$. $\square$

Let F be a number field. A character

$$\chi : \mathbf{A}_F^\times / F^\times \longrightarrow \mathbf{C}^\times$$

is called *algebraic* if for $\tau \in \mathrm{Hom}\,(F, \mathbf{C})$ there exist $m_\tau \in \mathbf{Z}$ such that

$$\chi|_{(F_\infty^\times)^0}(x) = \prod_{\tau \in \mathrm{Hom}\,(F,\mathbf{C})} \tau(x)^{-m_\tau}.$$

A set of integers $\{m_\tau\}$ arises from some algebraic character if and only if there is an integer d and a CM subfield $E \subset F$ such that if $\tau_1|_E = (\tau_2|_E) \circ c$ then $d = m_{\tau_1} + m_{\tau_2}$. For this and the proof of the next lemma see [Se1].

We will call a continuous character

$$\chi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbf{Q}}_l^\times$$

algebraic if it is de Rham at all places above l.

Lemma 4.1.3 *Let $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$. Let F be a number field. Let*

$$\chi : \mathbf{A}_F^\times / F^\times \longrightarrow \mathbf{C}^\times$$

be an algebraic character and for $\tau \in \mathrm{Hom}\,(F, \mathbf{C})$ let $m_\tau \in \mathbf{Z}$ satisfy

$$\chi|_{(F_\infty^\times)^0}(x) = \prod_{\tau \in \mathrm{Hom}\,(F,\mathbf{C})} \tau(x)^{-m_\tau}.$$

Then there is a continuous character

$$r_{l,\imath}(\chi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbf{Q}}_l^\times$$

with the following properties.

1. *For every prime $v \not| l$ of F we have*

$$r_{l,\imath}(\chi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)} = \chi_v \circ \mathrm{Art}_{F_v}^{-1}.$$

2. *If $v | l$ is a prime of F then $r_{l,\imath}(\chi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is potentially semistable (in fact potentially crystalline), and if χ_v is unramified then it is crystalline.*

3. If $v|l$ is a prime of F and if $\tau : F \hookrightarrow \overline{\mathbf{Q}}_l$ lies above v then

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i(r_{l,\imath}(\chi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$

unless $i = m_{\imath\tau}$ in which case

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i(r_{l,\imath}(\chi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

Any continuous algebraic character $\psi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbf{Q}}_l^\times$ arises in this way.

The character $r_{l,\imath}(\chi)$ is explicitly $\chi_{(l)} \circ \mathrm{Art}_F^{-1}$ where $\chi_{(l)} : \mathbf{A}_F^\times/\overline{F^\times(F_\infty^\times)^0} \to \overline{\mathbf{Q}}_l^\times$ is given by

$$\chi_{(l)}(x) = \left(\prod_{\tau \in \mathrm{Hom}\,(F,\mathbf{C})} (\imath^{-1}\tau)(x_l)^{-m_\tau} \right) \imath^{-1} \left(\left(\prod_{\tau \in \mathrm{Hom}\,(F,\mathbf{C})} \tau(x_\infty)^{m_\tau} \right) \chi(x) \right).$$

Lemma 4.1.4 *Let F be an imaginary CM field with maximal totally real subfield F^+. Let S be a finite set of primes of F^+ which split in F. Let I be a set of embeddings $F \hookrightarrow \mathbf{C}$ such that $I \coprod Ic$ is the set of all embeddings $F \hookrightarrow \mathbf{C}$. For $\tau \in I$ let m_τ be an integer. Suppose that*

$$\chi : \mathbf{A}_{F^+}^\times/(F^+)^\times \longrightarrow \mathbf{C}^\times$$

is algebraic, unramified at S and such that $\chi_v(-1)$ is independent of $v|\infty$. Then there is an algebraic character

$$\psi : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbf{C}^\times$$

which is unramified above S and satisfies

$$\psi \circ \mathbf{N}_{F/F^+} = \chi \circ \mathbf{N}_{F/F^+}$$

and

$$\psi|_{F_\infty^\times} = \prod_{\tau \in I} \tau^{m_\tau}(c\tau)^{w-m_\tau}$$

for some w.

Proof: From the discussion before lemma 4.1.3 we have that

$$\chi|_{((F_\infty^+)^\times)^0} = \prod_{\tau \in I} \tau^w$$

for some integer w. Choose an algebraic character

$$\phi : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbf{C}^\times$$

which is unramified above S and such that

$$\phi|_{F_\infty^\times} = \prod_{\tau \in I} \tau^{m_\tau}(c\tau)^{w-m_\tau}.$$

Replacing χ by $\chi\phi|^\times_{\mathbf{A}_F^+}$ we may suppose that χ has finite order and that $m_\tau = 0$ for all $\tau \in I$.

Let $U_S = \prod_{v \in S} \mathcal{O}_{F,v}^\times$ and $U_S^+ = \prod_{v \in S} \mathcal{O}_{F^+,v}^\times$. It suffices to prove that

$$\chi|_{(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S \overline{F^\times F_\infty^\times}} = 1.$$

If $\gamma_i \in F^\times$ and $x_i \in F_\infty^\times$ and $\gamma_i x_i$ tends to an element of $\mathbf{A}_{F^+}^\times U_S$, then for large i the ratio $\gamma_i^c/\gamma_i \in F^{\mathbf{N}_{F/F^+}=1}$ is a unit at all primes above S and tends to 1 in $(\mathbf{A}_F^{S,\infty})^\times$. As $\mathcal{O}_F^{N_{F/F^+}=1}$ is the group of roots of unity in F and hence is finite, we conclude that for i sufficiently large $\gamma_i^c/\gamma_i = 1$, i.e. $\gamma_i \in F^+$. Thus

$$(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S \overline{F^\times F_\infty^\times} = (\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times (F_\infty^+)^\times}.$$

We know that χ is trivial on $(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times ((F_\infty^+)^\times)^0}$.

Note that $\mathbf{A}_{F^+}^\times / (\mathbf{N}_{F/F^+}\mathbf{A}_F^\times)(F^+)^\times (F_\infty^+)^\times$ corresponds under the Artin map to the maximal quotient of $\mathrm{Gal}\,(F/F^+)$ in which all complex conjugations are trivial. Hence $\mathbf{A}_{F^+}^\times = (\mathbf{N}_{F/F^+}\mathbf{A}_F^\times)(F^+)^\times (F_\infty^+)^\times$ and we have an exact sequence

$$\begin{aligned}
&(0) \to ((\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times (F_\infty^+)^\times}) / ((\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times ((F_\infty^+)^\times)^0}) \\
&\to (F^+)^\times (\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) / U_S^+ \overline{(F^+)^\times ((F_\infty^+)^\times)^0} \to \mathbf{A}_{F^+}^\times / U_S^+ \overline{(F^+)^\times (F_\infty^+)^\times}) \to (0).
\end{aligned}$$

If M/F^+ denotes the maximal abelian extension unramified in S and if L/F^+ denotes the maximal totally real abelian extension unramified in S, then by class field theory this exact sequence corresponds to the exact sequence

$$(0) \to \mathrm{Gal}\,(M/LF) \to \mathrm{Gal}\,(M/F) \to \mathrm{Gal}\,(L/F^+) \to (0).$$

If $v|\infty$ write c_v for a complex conjugation at v. As $\mathrm{Gal}\,(M/LF)$ is generated by elements $c_{v_1}c_{v_2}$ where v_1 and v_2 are infinite places we see that the image of $((\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times (F_\infty^+)^\times}) / ((\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times ((F_\infty^+)^\times)^0})$ in $(F^+)^\times (\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) / U_S^+ \overline{(F^+)^\times ((F_\infty^+)^\times)^0}$ is generated by elements $(-1)_{v_1}(-1)_{v_2}$, where v_1 and v_2 are two infinite places. Thus χ will be trivial on the intersection $(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap U_S^+ \overline{(F^+)^\times (F_\infty^+)^\times}$ if and only if $\chi_{v_1}(-1)\chi_{v_2}(-1) = 1$ for all infinite places v_1 and v_2. The lemma follows. $\square$

Lemma 4.1.5 *Let F be an imaginary CM field with maximal totally real subfield F^+. Let I be a set of embeddings $F \hookrightarrow \overline{\mathbf{Q}}_l$ such that $I \coprod I^c$ is the set of all such embeddings. Choose an integer m_τ for all $\tau \in I$. Choose a finite set S of primes of F^+ which split in F and do not lie above l. Suppose that*

$$\chi : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \overline{\mathbf{Q}}_l^\times$$

is a continuous algebraic character which is unramified above S, crystalline at all primes above l and for which $\chi(c_v)$ is independent of the infinite place v of F^+. (Here c_v denotes complex conjugation at v.) Then there is a continuous algebraic character

$$\psi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbf{Q}}_l^\times$$

which is unramified above S and crystalline above l, such that

$$\psi\psi^c = \chi|_{\mathrm{Gal}\,(\overline{F}/F)},$$

and

$$\mathrm{gr}^{\,m_\tau}(\overline{\mathbf{Q}}_l(\psi) \otimes_{\tau, F_{v(\tau)}} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_{v(\tau)}/F_{v(\tau)})} \neq (0)$$

for all $\tau \in I$. (Here $v(\tau)$ is the place above l induced by τ.)

Proof: This is the Galois theoretic analogue of the previous lemma. It follows from lemmas 4.1.3 and 4.1.4. □

A slight variant on these lemmas is the following.

Lemma 4.1.6 *Suppose that $l > 2$ is a rational prime. Let F be an imaginary CM field with maximal totally real subfield F^+. Let S be a finite set of finite places of F containing all primes above l and satisfying $S^c = S$. Let*

$$\chi : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{O}_{\overline{\mathbf{Q}}_l}^\times$$

and

$$\overline{\theta} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbf{F}}_l^\times$$

be continuous characters with $\overline{\theta\theta}^c$ equal to the reduction of $\chi|_{\mathrm{Gal}\,(\overline{F}/F)}$. For $v \in S$, let

$$\psi_v : \mathrm{Gal}\,(\overline{F}_v/F_v) \longrightarrow \mathcal{O}_{\overline{\mathbf{Q}}_l}^\times$$

be a continuous character lifting $\overline{\theta}|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ such that

$$(\psi_v \psi_{v^c}^c)|_{I_{F_v}} = \chi|_{I_{F_v}}.$$

Suppose also that if $\tau : F \hookrightarrow \overline{\mathbf{Q}}_l$ lies above $v \in S$ then

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,m_\tau}(\psi_v \otimes_{\tau, F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1,$$

and that $m_\tau + m_{\tau\circ c}$ is independent of τ.

Then there is a continuous character

$$\theta : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \mathcal{O}_{\overline{\mathbf{Q}}_l}^\times$$

lifting $\overline{\theta}$ and such that

$$\theta\theta^c = \chi|_{\mathrm{Gal}\,(\overline{F}/F)}$$

and, for all $v \in S$,

$$\theta|_{I_{F_v}} = \psi|_{I_{F_v}}.$$

In particular θ is algebraic.

Proof: Choose an algebraic character ϕ of $\mathrm{Gal}\,(\overline{F}/F)$ such that if $\tau : F \hookrightarrow \overline{\mathbf{Q}}_l$ lies above $v \in S$ then

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{m_\tau}(\phi \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

Replace ψ_v by $\psi_v\phi|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}^{-1}$; $\overline{\theta}$ by $\overline{\theta}\phi^{-1}$; and χ by $\chi\phi_0^{-1}$, where ϕ_0 denotes ϕ composed with the transfer $\mathrm{Gal}\,(\overline{F}/F^+)^{\mathrm{ab}} \to \mathrm{Gal}\,(\overline{F}/F)^{\mathrm{ab}}$. Then we see that we may suppose that χ has finite image and each $\psi_v|_{I_{F_v}}$ has finite image.

Using the Artin map, think of χ as a character of $\mathbf{A}_{F^+}^\times/\overline{(F^+)^\times((F_\infty^+)^\times)^0}$; $\overline{\theta}$ as a character of $\mathbf{A}_F^\times/\overline{F^\times F_\infty^\times}$; and ψ_v as a character of $\mathcal{O}_{F,v}^\times$. Let $U_S = \prod_{v\in S}\mathcal{O}_{F,v}^\times$, $U_S^+ = \prod_{v\in S}\mathcal{O}_{F^+,v}^\times$ and $\psi = \prod_{v\in S}\psi_v : U_S \to \overline{\mathbf{Q}}_l^\times$. Note that $\psi|_{U_S^+} = \chi|_{U_S^+}$, that the reduction of χ equals $\overline{\theta}$ on $\mathbf{N}_{F/F^+}\mathbf{A}_F^\times$ and that the reduction of ψ equals $\overline{\theta}$ on U_S.

We get a character

$$\chi' = \chi\psi : U_S\mathbf{N}_{F/F^+}\mathbf{A}_F^\times/((U_S^+\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap \overline{(F^+)^\times((F_\infty^+)^\times)^0}) \longrightarrow \mathcal{O}_{\overline{\mathbf{Q}}_l}^\times.$$

The reduction of χ' equals the restriction of $\overline{\theta}$ to the domain of χ'. As in the proof of lemma 4.1.4 we see that

$$U_S(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap \overline{F^\times F_\infty^\times} = U_S^+(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap \overline{(F^+)^\times(F_\infty^+)^\times}.$$

However

$$(U_S^+(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap \overline{(F^+)^\times(F_\infty^+)^\times})/((U_S^+\mathbf{N}_{F/F^+}\mathbf{A}_F^\times) \cap \overline{(F^+)^\times((F_\infty^+)^\times)^0})$$

is a 2-group on which $\overline{\theta}$ vanishes. As $l > 2$ we see that χ' also vanishes on this group.

Extend χ' to a continuous character

$$\chi' : \mathbf{A}_F^\times/\overline{F^\times F_\infty^\times} \longrightarrow \overline{\mathbf{Q}}_l^\times$$

and let $\overline{\chi}'$ denote its reduction. Then $\overline{\theta}(\overline{\chi}')^{-1}$ is a continuous character

$$\mathbf{A}_F^\times/(U_S(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times)\overline{F^\times F_\infty^\times} \longrightarrow \overline{\mathbf{F}}_l^\times.$$

Lift it to a continuous character

$$\chi'' : \mathbf{A}_F^\times/(U_S(\mathbf{N}_{F/F^+}\mathbf{A}_F^\times)\overline{F^\times F_\infty^\times} \longrightarrow \overline{\mathbf{Q}}_l^\times.$$

Then $\theta = \chi'\chi''$ will suffice. □

4.2. *CM fields..* — Let F be a CM field. By a *RACSDC* (regular, algebraic, conjugate self dual, cuspidal) automorphic representation π of $GL_n(\mathbf{A}_F)$ we mean a cuspidal automorphic representation such that

– $\pi^\vee \cong \pi^c$, and
– π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbf{Q}$ of GL_n.

Let $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\mathbf{C})}$ satisfy

– $a_{\tau,1} \geq ... \geq a_{\tau,n}$, and
– $a_{\tau c,i} = -a_{\tau,n+1-i}$.

Let Ξ_a denote the irreducible algebraic representation of $GL_n^{\mathrm{Hom}\,(F,\mathbf{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RACSDC automorphic representation π of $GL_n(\mathbf{A}_F)$ has *weight* a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$.

Let S be a finite set of finite places of F. For $v \in S$ let ρ_v be an irreducible square integrable representation of $GL_n(F_v)$. We will say that a RACSDC automorphic representation π of $GL_n(\mathbf{A}_F)$ has *type* $\{\rho_v\}_{v\in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

The following is a restatement of theorem VII.1.9 of [HT].

Proposition 4.2.1 *Let $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$. Let F be an imaginary CM field, S a finite non-empty set of finite places of F and, for $v \in S$, ρ_v a square integrable representation of $GL_n(F_v)$. Let $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\mathbf{C})}$ be as above. Suppose that π is a RACSDC automorphic representation of $GL_n(\mathbf{A}_F)$ of weight a and type $\{\rho_v\}_{v\in S}$. Then there is a continuous semisimple representation*

$$r_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

with the following properties.

1. For every prime $v \not| l$ of F we have

$$r_{l,\imath}(\pi)|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v/F_v)} = r_l(\imath^{-1}\pi_v)^\vee(1-n)^{\mathrm{ss}}.$$

2. $r_{l,\imath}(\pi)^c = r_{l,\imath}(\pi)^\vee \epsilon^{1-n}$.

3. If $v|l$ *is a prime of* F *then* $r_{l,\imath}(\pi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ *is potentially semistable, and if* π_v *is unramified then it is crystalline.*

4. If $v|l$ *is a prime of* F *and if* $\tau : F \hookrightarrow \overline{\mathbf{Q}}_l$ *lies above* v *then*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i (r_{l,\imath}(\pi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$

unless $i = a_{\imath\tau,j} + n - j$ *for some* $j = 1, ..., n$ *in which case*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i (r_{l,\imath}(\pi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

Moreover if $\psi : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$ *is an algebraic character satisfying* $\psi \circ c = \psi^{-1}$ *then*

$$r_{l,\imath}(\pi \otimes (\psi \circ \det)) = r_{l,\imath}(\pi) \otimes r_{l,\imath}(\psi).$$

Proof: We can take $r_{l,\imath}(\pi) = R_l(\pi^\vee)(1-n)$ in the notation of [HT]. Note that the definition of highest weight we use here differs from that in [HT]. $\square$

The representation $r_{l,\imath}(\pi)$ can be taken to be valued in $GL_n(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbf{Q}_l$. Thus we can reduce it modulo the maximal ideal of $\mathcal{O}$ and semisimplify to obtain a continuous semisimple representation

$$\overline{r}_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{F}}_l)$$

which is independent of the choices made. Note that if $r_{l,\imath}(\pi)$ (resp. $\overline{r}_{l,\imath}(\pi)$) is irreducible it extends to a continuous homomorphism

$$r_{l,\imath}(\pi)' : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\overline{\mathbf{Q}}_l)$$

(resp.

$$\overline{r}_{l,\imath}(\pi)' : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\overline{\mathbf{F}}_l)).$$

Let $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$. Suppose that $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)}$ satisfies

– $a_{\tau,1} \geq ... \geq a_{\tau,n}$, and

– $a_{\tau c,i} = -a_{\tau,n+1-i}$.

Then we define $\imath_* a$ by

$$(\imath_* a)_{\imath\tau,i} = a_{\tau,i}.$$

Suppose that $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)}$ satisfies the conditions of the previous paragraph, that S is a finite set of finite places of F not containing any prime above l and that ρ_v is a discrete series representation of $GL_n(F_v)$ over $\overline{\mathbf{Q}}_l$ for all $v \in S$. We will call a continuous semisimple representation

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

(resp.

$$\bar{r} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{F}}_l))$$

automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if there is an isomorphism $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ and a RACSDC automorphic representation π of $GL_n(\mathbf{A}_F)$ of weight $\imath_* a$ and type $\{\imath\rho_v\}_{v\in S}$ (resp. and with π_l unramified) such that $r \cong r_{l,\imath}(\pi)$ (resp. $\bar{r} \cong \bar{r}_{l,\imath}(\pi)$). We will say that r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and *level prime to l* if there is an isomorphism $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ and a RACSDC automorphic representation π of $GL_n(\mathbf{A}_F)$ of weight $\imath_* a$ and type $\{\imath\rho_v\}_{v\in S}$ and with π_l unramified such that $r \cong r_{l,\imath}(\pi)$.

The following lemma is standard.

Lemma 4.2.2 *Suppose that E/F is a soluble Galois extension of CM fields. Suppose that*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

is a continuous semisimple representation and that $r|_{\mathrm{Gal}\,(\overline{F}/E)}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$. Let S_F denote the set of places of F which lie under an element of S. Then we have the following.

1. $a_\tau = a_{\tau'}$ if $\tau|_F = \tau'|_F$ so we can define a_F by $a_{F,\sigma} = a_{\widetilde{\sigma}}$ for any extension $\widetilde{\sigma}$ of σ to E.

2. r is automorphic over F of weight a_F and type $\{\rho'_v\}_{v\in S_F}$ for some square integrable representations ρ'_v.

Proof: Inductively we may reduce to the case that E/F is cyclic of prime order. Suppose that $\mathrm{Gal}\,(E/F) = \langle\sigma\rangle$ and that $r = r_{l,\imath}(\pi)$, for π a RACSDC automorphic representation of $GL_n(\mathbf{A}_E)$ of weight a and level $\{\rho_v\}_{v\in S}$. Then $r|^{\sigma}_{\mathrm{Gal}\,(\overline{F}/E)} \cong r|_{\mathrm{Gal}\,(\overline{F}/E)}$ so that $\pi^\sigma = \pi$. By theorem 4.2 of [AC] π descends to a RACSDC automorphic representation π_F of $GL_n(\mathbf{A}_F)$. As r and $r_{l,\imath}(\pi_F)$ are irreducible and have the same restriction to $\mathrm{Gal}\,(\overline{F}/E)$ we see that $r = r_{l,\imath}(\pi_F) \otimes \chi = r_{l,\imath}(\pi_F \otimes (\chi \circ \mathrm{Art}_F))$ for some character χ of $\mathrm{Gal}\,(E/F)$. The lemma follows. $\Box$

4.3. *Totally real fields.* — Now let F^+ denote a totally real field. By a *RAESDC* (regular, algebraic, essentially self dual, cuspidal) automorphic representation π of $GL_n(\mathbf{A}_{F^+})$ we mean a cuspidal automorphic representation such that

– $\pi^\vee \cong \chi\pi$ for some character $\chi : (F^+)^\times\backslash\mathbf{A}^\times_{F^+} \to \mathbf{C}^\times$ with $\chi_v(-1)$ independent of $v|\infty$, and

– π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F^+ to $\mathbf{Q}$ of GL_n.

One can ask whether if these conditions are met for some $\chi : (F^+)^\times \backslash \mathbf{A}_{F^+}^\times \to \mathbf{C}^\times$, they will automatically be met for some such χ' with $\chi'_v(-1)$ independent of $v|\infty$. This is certainly true if n is odd. (As then χ^n is a square, so that $\chi_v(-1) = 1$ for all $v|\infty$.) It is also true if $n = 2$ (As in this case we can take χ to be the inverse of the central character of π and the parity condition is equivalent to the fact that if a holomorphic Hilbert modular form has weight $(k_\tau)_{\tau \in \mathrm{Hom}\,(F^+,\mathbf{R})}$ then $k_\tau \bmod 2$ is independent of τ.)

Let $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F^+,\mathbf{C})}$ satisfy

$$a_{\tau,1} \geq ... \geq a_{\tau,n}$$

Let Ξ_a denote the irreducible algebraic representation of $GL_n^{\mathrm{Hom}\,(F^+,\mathbf{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RAESDC automorphic representation π of $GL_n(\mathbf{A}_F)$ has *weight* a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$. In that case there is an integer w_a such that

$$a_{\tau,i} + a_{\tau,n+1-i} = w_a$$

for all $\tau \in \mathrm{Hom}\,(F^+, \mathbf{C})$ and all $i = 1, ..., n$.

Let S be a finite set of finite places of F^+. For $v \in S$ let ρ_v be an irreducible square integrable representation of $GL_n(F_v^+)$. We will say that a RAESDC automorphic representation π of $GL_n(\mathbf{A}_{F^+})$ has *type* $\{\rho_v\}_{v\in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

Proposition 4.3.1 *Let $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$. Let F^+ be a totally real field, S a finite non-empty set of finite places of F^+ and, for $v \in S$, ρ_v a square integrable representation of $GL_n(F_v^+)$. Let $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F^+,\mathbf{C})}$ be as above. Suppose that π is a RAESDC automorphic representation of $GL_n(\mathbf{A}_{F^+})$ of weight a and type $\{\rho_v\}_{v\in S}$. Specifically suppose that $\pi^\vee \cong \pi\chi$ where $\chi : \mathbf{A}_{F^+}^\times/(F^+)^\times \to \mathbf{C}^\times$. Then there is a continuous semisimple representation*

$$r_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

with the following properties.

1. For every prime $v \not| l$ of F^+ we have

$$r_{l,\imath}(\pi)|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = r_l(\imath^{-1}\pi_v)^\vee(1-n)^{\mathrm{ss}}.$$

2. $r_{l,\imath}(\pi)^\vee = r_{l,\imath}(\pi)\epsilon^{n-1}r_{l,\imath}(\chi)$.

3. If $v|l$ is a prime of F^+ then the restriction $r_{l,\imath}(\pi)|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ is potentially semistable, and if π_v is unramified then it is crystalline.

4. If $v|l$ is a prime of F^+ and if $\tau : F^+ \hookrightarrow \overline{\mathbf{Q}}_l$ lies above v then

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i (r_{l,\imath}(\pi) \otimes_{\tau, F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 0$$

unless $i = a_{\imath\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i (r_{l,\imath}(\pi) \otimes_{\tau, F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 1.$$

Moreover if $\psi : \mathbf{A}_{F^+}^\times/(F^+)^\times \to \mathbf{C}^\times$ is an algebraic character then

$$r_{l,\imath}(\pi \otimes (\psi \circ \det)) = r_{l,\imath}(\pi) \otimes r_{l,\imath}(\psi).$$

Proof: Let F be an imaginary CM field with maximal totally real subfield F^+, such that all primes above l and all primes in S split in F/F^+. Choose an algebraic character $\phi : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ such that $\chi \circ \mathbf{N}_{F/F^+} = \phi \circ \mathbf{N}_{F/F^+}$. (See lemma 4.1.4.) Let π_F denote the base change of π to F. Applying proposition 4.2.1 to $\pi_F \phi$, we obtain a continuous semi-simple representation

$$r_F : \mathrm{Gal}\,(\overline{F}^+/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

such that for every prime $v \not| l$ of F we have

$$r_F|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v^+/F_v)} = r_l(\imath^{-1}\pi_{v|_{F^+}})^\vee(1-n)|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v^+/F_v)}.$$

Letting the field F vary we can piece together the representations r_F to obtain r. (See the argument of the second half of the proof of theorem VII.1.9 of [HT].) $\square$

The representation $r_{l,\imath}(\pi)$ can be taken to be valued in $GL_n(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbf{Q}_l$. Thus we can reduce it modulo the maximal ideal of $\mathcal{O}$ and semisimplify to obtain a continuous semisimple representation

$$\overline{r}_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{F}}_l)$$

which is independent of the choices made.

Let $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$. Suppose that $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)}$ satisfies

$$a_{\tau,1} \geq ... \geq a_{\tau,n}.$$

Then we define $\imath_* a$ by

$$(\imath_* a)_{\imath\tau,i} = a_{\tau,i}.$$

Suppose that $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)}$ satisfies the conditions of the previous paragraph, that S is a finite set of finite places of F^+ not containing any prime above l and that ρ_v is a discrete series representation of $GL_n(F_v^+)$ over $\overline{\mathbf{Q}}_l$ for all $v \in S$. We will call a continuous semisimple representation

$$r : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

(resp.

$$\overline{r} : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{F}}_l))$$

automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if there is an isomorphism $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ and a RAESDC automorphic representation π of $GL_n(\mathbf{A}_{F^+})$ of weight $\imath_* a$ and type $\{\imath\rho_v\}_{v\in S}$ (resp. and with π_l unramified) such that $r \cong r_{l,\imath}(\pi)$ (resp. $\overline{r} \cong \overline{r}_{l,\imath}(\pi)$). We will say that r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and *level prime to l* if there is an isomorphism $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ and a RAESDC automorphic representation π of $GL_n(\mathbf{A}_{F^+})$ of weight $\imath_* a$ and type $\{\imath\rho_v\}_{v\in S}$ and with π_l unramified such that $r \cong r_{l,\imath}(\pi)$.

The following two lemmas are proved just as lemma 4.2.2.

Lemma 4.3.2 *Let E^+/F^+ be a soluble Galois extension of CM fields. Suppose that*

$$r : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

is a continuous semisimple representation and that $r|_{\mathrm{Gal}\,(\overline{F}^+/E^+)}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$. Let S_{F^+} denote the set of places of F^+ under an element of S. Then we have the following.

1. *$a_\tau = a_{\tau'}$ if $\tau|_{F^+} = \tau'|_{F^+}$ so we can define a_{F^+} by $a_{F^+,\sigma} = a_{\widetilde{\sigma}}$ for any extension $\widetilde{\sigma}$ of σ to E^+.*
2. *r is automorphic over F^+ of weight a_{F^+} and type $\{\rho'_v\}_{v\in S_{F^+}}$ for some square integrable representations ρ'_v.*

Lemma 4.3.3 *Let F be a CM field with maximal totally real subfield F^+. Suppose that $\psi : \mathrm{Gal}\,(\overline{F}/F) \to \overline{\mathbf{Q}}_l^\times$ is a continuous algebraic character and that*

$$r : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

is a continuous semisimple representation and that $r|_{\mathrm{Gal}\,(\overline{F}/F)} \otimes \psi$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$. Let S_{F^+} denote the set of places of F^+ under an element of S. Then r is automorphic over F^+ of weight b and type $\{\rho'_v\}_{v\in S_{F^+}}$ for some square integrable representations ρ'_v and for some b. Moreover, for all $\tau : F \hookrightarrow \overline{\mathbf{Q}}_l$ and all $i = 1, ..., n$, the co-ordinate $a_{\tau,i}$ equals $b_{\tau|_{F^+},i}$ plus the unique number j such that $\mathrm{gr}^j(\psi \otimes_{\tau,F_v} B_{\mathrm{DR}}) \neq (0)$ (where v is the place of F induced by τ).

4.4. *Modularity lifting theorems: the minimal case.* — In this section we use base change to translate theorem 3.5.1 into a modularity lifting theorem on $GL(n)$. The results here are entirely superseded by the results of [Tay] and for the reader interested only in the main results of [Tay] and [HSBT] this section could be skipped.

We start with a lemma about congruences which is analogous to a trick invented by Skinner and Wiles in the case of GL_2, see [SW].

Lemma 4.4.1 *Let F^+ be a totally real field of even degree and E an imaginary quadratic field such that $F = F^+E/F^+$ is unramified at all finite primes. Let $n \in \mathbf{Z}_{\geq 2}$ and let $l > n$ be a prime which splits in E. Let $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ and let S_l denote the set of primes of F above l. Let π be a RACSDC automorphic representation of $GL_n(\mathbf{A}_F)$ of weight a and type $\{\rho_v\}_{v\in S}$ where S is a finite non-empty set of primes split over F^+. Assume that $4|\#(S \cup S^c)$. Suppose that π_v is unramified if v is not split over F^+ or if $v|l$. Let R be a finite set of primes of F such that if $v \in R$ then*

- *$v \notin S \cup S^c \cup S_l$,*
- *v is split over F^+,*
- *$\mathbf{N}v \equiv 1 \bmod l$,*
- *$\pi_v^{\mathrm{Iw}(v)} \neq (0)$.*

Let S_a be a non-empty finite set of primes of F such that $S_a = S_a^c$ and $S_a \cap (R \cup S \cup S_l) = \emptyset$.

Then there is a RACSDC automorphic representation π' of $GL_n(\mathbf{A}_F)$ of weight a and type $\{\rho_v\}_{v\in S}$ with the following properties:

- *$\overline{r}_{l,\imath}(\pi) \cong \overline{r}_{l,\imath}(\pi')$;*
- *if $v \notin S_a$ and π_v is unramified then π'_v is unramified;*
- *if v in R then $r_l(\pi'_v)^\vee(1-n)(I_{F_v})$ is finite.*

Proof: Let $S(B)$ denote the set of primes of F^+ below an element of S. Choose B and $\ddagger$ as at the start of section 3.3. These define an algebraic group G. Consider open compact subgroups $U = \prod_v U_v$ of $G(\mathbf{A}_{F^+}^\infty)$ where

– if v is inert in F, then U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$;

– if v is split in F and v lies below S then $U_v = G(\mathcal{O}_{F^+,v})$;

– if v does not lie below $R \cup S_a$, if v is split in F and if π_v is unramified then $U_v = G(\mathcal{O}_{F^+,v})$;

– if v lies below R and if w is a prime of F above v then $U_v = i_w^{-1}\mathrm{Iw}(w)$;

– if v lies below S_a then U_v contains only one element of finite order, namely 1.

We now apply lemma 3.4.1 with $\chi_v = 1$ and $\chi'_v = \prod_{i=1}^n \chi'_{v,i}$ for all $v \in R$, where we choose $\chi'_{v,i}$ each of l-power order and with $\chi'_{v,i} \neq \chi'_{v,j}$ for $i \neq j$. (This is possible as $l > n$.) The lemma then follows from lemma 3.1.6 and proposition 3.3.2. (The fact that the $\chi'_{v,i}$ are distinct gives the finiteness of the image of intertia at v.) $\Box$

Next we prove a 'minimal' modularity lifting theorem over a CM field.

Theorem 4.4.2 *Let F be an imaginary CM field and let F^+ denote its maximal totally real subfield. Let $n \in \mathbf{Z}_{\geq 1}$ and let $l > n$ be a prime which is unramified in F. Let*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r.

1. *$r^c \cong r^\vee \epsilon^{1-n}$.*
2. *r is unramified at all but finitely many primes.*
3. *For all places $v|l$ of F, $r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is crystalline.*
4. *There is an element $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)}$ such that*
   - *for all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbf{Q}}_l)$ we have*

$$l - 1 - n \geq a_{\tau,1} \geq ... \geq a_{\tau,n} \geq 0$$

or

$$l - 1 - n \geq a_{\tau c,1} \geq ... \geq a_{\tau c,n} \geq 0;$$

   - *for all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbf{Q}}_l)$ and all $i = 1, ..., n$*

$$a_{\tau c,i} = -a_{\tau,n+1-i};$$

   - *for all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbf{Q}}_l)$ above a prime $v|l$ of F,*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i (r \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$

unless $i = a_{\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^i (r \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

5. *There is a non-empty finite set S of places of F not dividing l and for each $v \in S$ a square integrable representation ρ_v of $GL_n(F_v)$ over $\overline{\mathbf{Q}}_l$ such that*

$$r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v/F_v)} = r_l(\rho_v)^\vee(1-n)^{\mathrm{ss}}.$$

If $\rho_v = \mathrm{Sp}_{m_v}(\rho'_v)$ then set

$$\widetilde{r}_v = r_l((\rho'_v)^\vee|\ |^{(n/m_v-1)(1-m_v)/2}).$$

Note that $r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ *has a unique filtration* Fil_v^j *such that*

$$\mathrm{gr}_v^j r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)} \cong \widetilde{r}_v \epsilon^j$$

for $j = 0, ..., m_v - 1$ *and equals* (0) *otherwise. We assume that* $\widetilde{r}_v$ *has irreducible reduction* $\overline{r}_v$*. Then* $\overline{r}|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ *inherits a filtration* $\overline{\mathrm{Fil}}_v^j$ *with*

$$\overline{\mathrm{gr}}_v^j \overline{r}|_{\mathrm{Gal}\,(\overline{F}_v/F_v)} \cong \overline{r}_v \epsilon^j$$

for $j = 0, ..., m_v - 1$*. Finally we suppose that for* $j = 1, ..., m_v$ *we have*

$$\overline{r}_v \not\cong \overline{r}_v \epsilon^i.$$

6. *For all finite places* $v \not| l$ *with* $v \notin S \cup S^c$ *the image* $r(I_{F_v})$ *is finite.*
7. $\overline{F}^{\ker \mathrm{ad}\, \overline{r}}$ *does not contain* $F(\zeta_l)$*.*
8. *The image* $\overline{r}(\mathrm{Gal}\,(\overline{F}/F(\zeta_l)))$ *is big in the sense of definition 2.5.1.*
9. *The representation* $\overline{r}$ *is irreducible and automorphic of weight* a *and type* $\{\rho_v\}_{v \in S}$ *with* $S \neq \emptyset$*.*

Then r *is automorphic of weight* a *and type* $\{\rho_v\}_{v\in S}$ *and level prime to* l*.*

Proof: Suppose that $\overline{r} = \overline{r}_{l,\imath}(\pi)$, where $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ and where π is a RACSDC automorphic representation of $GL_n(\mathbf{A}_F)$ of weight $\imath_* a$ and type $\{\imath\rho_v\}_{v\in S}$ and with π_l unramified. Let S_l denote the primes of F above l. Let R denote the primes of F outside $S^c \cup S \cup S_l$ at which r or π is ramified. Because $\overline{F}^{\ker \mathrm{ad}\,\overline{r}}$ does not contain $F(\zeta_l)$, we can choose a prime v_1 of F with the following properties

- $v_1 \notin R \cup S_l \cup S \cup S^c$,
- v_1 is unramified over a rational prime p for which $[F(\zeta_p) : F] > n$,
- v_1 does not split completely in $F(\zeta_l)$,
- $\mathrm{ad}\,\overline{r}(\mathrm{Frob}_{v_1}) = 1$.

(We will use primes above v_1 as auxiliary primes to augment the level so that the open compact subgroups of the finite adelic points of certain unitary groups we consider will be sufficiently small. The properties of v_1 will ensure that the Galois deformation problems we consider will not change when we allow ramification at primes above v_1.)

Choose a CM field L/F with the following properties

- $L = L^+E$ with E an imaginary quadratic field and L^+ totally real.
- $4|[L^+ : F^+]$.
- L/F is Galois and soluble.
- L is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F.

– L/L^+ is everywhere unramified.
– l splits in E and is unramified in L.
– v_1 and v_1^c split completely in L/F and in L/L^+.
– All primes in S split completely in L/F and in L/L^+.
– Let π_L denote the base change of π to L. If v is a prime of L not lying above $S \cup S^c$ then $\pi_v^{\mathrm{Iw}(v)} \neq (0)$.
– If v is a place of L above R then $r|_{\mathrm{Gal}(\overline{L}/L)}$ is unramified at v.

Let $S(L)$ (resp. $S_l(L)$) denote the set of places of L above S (resp. l). Let $a_L \in (\mathbf{Z}^n)^{\mathrm{Hom}(L,\overline{\mathbf{Q}}_l)}$ be defined by $a_{L,\tau} = a_{\tau|_F}$. By theorem 4.2 of [AC] we know that $\overline{r}|_{\mathrm{Gal}(\overline{F}/L)}$ is automorphic of weight a_L and type $\{\rho_{v|_F}\}_{v \in S(L)}$. (The base change must be cuspidal as it is square integrable at finite places in S.) By lemma 4.4.1 there is a RACSDC automorphic representation π' of $GL_n(\mathbf{A}_L)$ of weight a_L and type $\{\rho_{v|_F}\}_{v \in S(L)}$ and level prime to l such that

– $\overline{r}|_{\mathrm{Gal}(\overline{F}/L)} = \overline{r}_{l,\imath}(\pi')$, and
– $r_{l,\imath}(\pi')$ is finitely ramified at all primes outside $S(L) \cup S(L)^c \cup S_l(L)$.

(If $v|v_1$ or v_1^c then $\overline{r}_{l,\imath}(\pi')$ is unramified at v and all the eigenvalues of the matrix $\overline{r}_{l,\imath}(\pi')(\mathrm{Frob}_v)$ are equal. As $\mathbf{N}v \not\equiv 1 \bmod l$ we see that $r_{l,\imath}(\pi')$ is finitely ramified at v.)

Choose a decomposition $S_l(L) = \widetilde{S}_l(L) \coprod \widetilde{S}_l(L)^c$. Also choose an algebraic character $\psi : \mathbf{A}_L^\times / L^\times \to \mathbf{C}^\times$ such that

– $\psi \circ \mathbf{N}_{F/F^+} = 1$;
– ψ is unramified at $S_l(L) \cup S(L)$; and
– $\pi' \otimes \psi$ has weight $\imath_* a'$ where

$$l - 1 - n \geq a'_{\tau,1} \geq ... \geq a'_{\tau,n} \geq 0$$

for all $\tau : L \hookrightarrow \overline{\mathbf{Q}}_l$ lying over an element of $\widetilde{S}_l(L)$.

(This is possible by lemma 4.1.4. The point of this step is to arrange that for each place $v|l$ of F the weights $a'_{\tau,i}$ for τ above v are all in the *same* range of length $l-1-n$. This was assumed in theorem 3.5.1, so as we could apply Fontaine-Laffaille theory to calculate the local deformation ring, see section 2.4.1.)

Choose a CM field M/L with the following properties.

– M/L is Galois and soluble.
– M is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over L.
– l is unramified in M.
– v_1 splits completely in M/F.
– All primes in S split completely in M/L.

– Let $(\pi' \otimes \psi)_M$ denote the base change of $\pi' \otimes (\psi \circ \det)$ to M. If v is a prime of M not lying above $S \cup S^c$ then $(\pi' \otimes \psi)_{M,v}$ is unramified.

– If v is a place of M not lying above $S(L) \cup S(L)^c \cup S_l(L)$ then $(r \otimes r_{l,\imath}(\psi))|_{\mathrm{Gal}\,(\overline{L}/L)}$ is unramified at v.

Let $S(M)$ denote the set of places of M above S. Let $a'_M \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(M,\overline{\mathbf{Q}}_l)}$ be defined by $a'_{M,\tau} = a'_{\tau|_F}$. Let $S(M^+)$ denote the set of places of M^+ below an element of $S(M)$. Then $\#S(M^+)$ is even and every element of $S(M^+)$ splits in M. Choose a division algebra B/M and an involution $\ddagger$ of B as at the start of section 3.3, with $S(B) = S(M^+)$. Let $S_l(M^+)$ denote the primes of M^+ above l and let $S_a(M^+)$ denote the primes of M^+ above $v_1|_{F^+}$. Let $T(M^+) = S(M^+) \cup S_l(M^+) \cup S_a(M^+)$. It follows from proposition 3.3.2 and theorem 3.5.1 that $r|_{\mathrm{Gal}\,(\overline{F}/M)} \otimes r_{l,\imath}(\psi)|_{\mathrm{Gal}\,(\overline{F}/M)}$ is automorphic of weight a'_M and type $\{\rho_{v|_F}\}_{v \in S(M)}$. The theorem now follows from lemma 4.2.2. $\Box$

Let us say a few words about the conditions in this theorem. The first condition ensures that r is conjugate self-dual. Only for such representations will the numerology behind the Taylor-Wiles argument work. Also it is only for such representations that one can work on a unitary group. Indeed whenever one has a cuspidal automorphic representation of $GL_n(\mathbf{A}_F)$ for which one knows how to construct a Galois representation, that Galois representation will have this property. The second condition should be necessary, i.e. it should hold for any Galois representation associated to an automorphic form. A weakened form of the third condition which required only that these restrictions are de Rham is also expected to be necessary. The stronger form here (requiring the restrictions to be crystalline), the assumption that l is unramified in F and the bounds on the Hodge-Tate numbers in condition four are all needed so that we can apply the theory of Fontaine and Laffaille to calculate the local deformation rings at primes above l. Condition four also requires the Hodge-Tate numbers to be distinct. Otherwise the numerology behind the Taylor-Wiles method would fail. The fifth condition is there to ensure that the corresponding automorphic form will be discrete series at some places (ie those in S). With the current state of the trace formula this is necessary to move automorphic forms between unitary groups and GL_n and also to construct Galois representations for automorphic forms on GL_n. The exact form of condition five is also designed to also make the deformation problem at the places $v \in S$ well behaved. The sixth condition is designed so that we can use base change to put us in a situation where we can apply a *minimal* $R = \mathbf{T}$ theorem. In chapter 4 we will show that a conjecture about $\mathrm{mod}\, l$ automorphic forms on unitary groups which we call “Ihara's lemma” implies that we could remove this condition. The seventh condition is to allow us to choose auxiliary primes which can be used to augment the level and ensure that certain level structures we work with are sufficiently small. The eighth condition is to make the Cebotarev argument used in the Taylor-Wiles argument work. It seems to be often satisfied in practice.

Now we turn to the case of a totally real field.

Theorem 4.4.3 *Let F^+ be a totally real field. Let $n \in \mathbf{Z}_{\geq 1}$ and let $l > n$ be a prime which is unramified in F^+. Let*

$$r : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r.

1. *$r^\vee \cong r\epsilon^{n-1}\chi$ for some character $\chi : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow \overline{\mathbf{Q}}_l^\times$ with $\chi(c_v)$ independent of $v|\infty$. (Here c_v denotes a complex conjugation at v.)*
2. *r ramifies at only finitely many primes.*
3. *For all places $v|l$ of F^+, $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ is crystalline.*
4. *There is an element $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F^+,\overline{\mathbf{Q}}_l)}$ such that*
   – *for all $\tau \in \mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)$ we have*

$$l - 1 - n + a_{\tau,n} \geq a_{\tau,1} \geq \ldots \geq a_{\tau,n};$$

   – *for all $\tau \in \mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)$ above a prime $v|l$ of F^+,*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 0$$

unless $i = a_{\tau,j} + n - j$ for some $j = 1, \ldots, n$ in which case

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 1.$$

5. *There is a finite non-empty set S of places of F^+ not dividing l and for each $v \in S$ a square integrable representation ρ_v of $GL_n(F_v^+)$ over $\overline{\mathbf{Q}}_l$ such that*

$$r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = r_l(\rho_v)^\vee(1-n)^{\mathrm{ss}}.$$

If $\rho_v = \mathrm{Sp}_{\,m_v}(\rho'_v)$ then set

$$\widetilde{r}_v = r_l((\rho'_v)^\vee|\ |^{(n/m_v-1)(1-m_v)/2}).$$

Note that $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ has a unique filtration Fil_v^j such that

$$\mathrm{gr}_{\,v}^{\,j} r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} \cong \widetilde{r}_v\epsilon^j$$

for $j = 0, ..., m_v - 1$ *and equals* (0) *otherwise. We assume that* $\widetilde{r}_v$ *has irreducible reduction* $\overline{r}_v$ *such that*

$$\overline{r}_v \not\cong \overline{r}_v \epsilon^j$$

for $j = 1, ..., m_v$. *Then* $\overline{r}|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ *inherits a unique filtration* $\overline{\mathrm{Fil}}_v^j$ *with*

$$\overline{\mathrm{gr}}_v^j \overline{r}|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} \cong \overline{r}_v \epsilon^j$$

for $j = 0, ..., m_v - 1$.

6. *If* $v \notin S$ *and* $v \not| l$ *then* $r(I_{F_v^+})$ *is finite.*
7. $(\overline{F}^+)^{\ker \mathrm{ad}\, \overline{r}}$ *does not contain* $F^+(\zeta_l)$.
8. *The image* $\overline{r}(\mathrm{Gal}\,(\overline{F}^+/F^+(\zeta_l)))$ *is big in the sense of definition 2.5.1.*
9. $\overline{r}$ *is irreducible and automorphic of weight* a *and type* $\{\rho_v\}_{v \in S}$ *with* $S \neq \emptyset$.

Then r *is automorphic of weight* a *and type* $\{\rho_v\}_{v \in S}$ *and level prime to* l.

Proof: Choose an imaginary CM field F with maximal totally real subfield F^+ such that

- all primes above l split in F/F^+,
- all primes in S split in F/F^+, and
- F is linearly disjoint from $(\overline{F}^+)^{\ker \overline{r}}(\zeta_l)$ over F^+.

Choose an algebraic character

$$\psi : \mathrm{Gal}\,(\overline{F}^+/F) \longrightarrow \overline{\mathbf{Q}}_l^\times$$

such that

- $\chi|_{\mathrm{Gal}\,(\overline{F}^+/F)} = \psi\psi^c$,
- ψ is unramified above S,
- ψ is crystalline above l, and
- for each $\tau : F^+ \hookrightarrow \overline{\mathbf{Q}}_l$ there exists an extension $\widetilde{\tau} : F \hookrightarrow \overline{\mathbf{Q}}_l$ such that

$$\mathrm{gr}^{-a_{\tau,n}}(\overline{\mathbf{Q}}_l(\psi) \otimes_{\widetilde{\tau}, F_{v(\widetilde{\tau})}} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_{v(\widetilde{\tau})}/F_{v(\widetilde{\tau})})} \neq (0),$$

where $v(\widetilde{\tau})$ is the place of F above l determined by $\widetilde{\tau}$.

(This is possible by lemma 4.1.5.) Now apply theorem 4.4.2 to $r|_{\mathrm{Gal}\,(\overline{F}^+/F)}\psi$ and this theorem follows from lemma 4.3.3. □

As the conditions of this theorem are a bit complicated we give a special case as a corollary.

Corollary 4.4.4 *Let $n \in \mathbf{Z}_{\geq 1}$ be even and let $l > \max\{3, n\}$ be a prime. Let S be a finite non-empty set of primes such that if $q \in S$ then $q \neq l$ and $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n$. Let*

$$r : \mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow GSp_n(\mathbf{Z}_l)$$

be a continuous irreducible representation with the following properties.

1. *r ramifies at only finitely many primes.*
2. *$r|_{\mathrm{Gal}\,(\overline{\mathbf{Q}}_l/\mathbf{Q}_l)}$ is crystalline and $\dim_{\mathbf{Q}_l} \mathrm{gr}^{\,i}(r \otimes_{\mathbf{Q}_l} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{\mathbf{Q}}_l/\mathbf{Q}_l)} = 0$ unless $i \in \{0, 1, ..., n-1\}$, in which case it has dimension 1.*
3. *If $q \in S$ then $r|^{\mathrm{ss}}_{G_{\mathbf{Q}_q}}$ is unramified and $r|^{\mathrm{ss}}_{G_{\mathbf{Q}_q}}(\mathrm{Frob}_q)$ has eigenvalues $\{\alpha q^i : \ i = 0, 1, ..., n-1\}$ for some α.*
4. *If $q \notin S \cup \{l\}$ then $r(I_{\mathbf{Q}_q})$ is finite.*
5. *The image of $r \bmod l$ contains $Sp_n(\mathbf{F}_l)$.*
6. *$r \bmod l$ is automorphic of weight 0 and type $\{\mathrm{Sp}\,_n(1)\}_{q \in S}$.*

Then r is automorphic of weight 0 and type $\{\mathrm{Sp}\,_n(1)\}_{\{q\}}$ and level prime to l.

Proof: Let $\overline{r} = r \bmod l$. As $PSp_n(\mathbf{F}_l)$ is simple, the maximal abelian quotient of $\mathrm{ad}\,\overline{r}(G_{\mathbf{Q}})$ is

$$\overline{r}(G_{\mathbf{Q}})/(\overline{r}(G_{\mathbf{Q}}) \cap \mathbf{F}_l^\times)Sp_n(\mathbf{F}_l) \subset PGSp_n(\mathbf{F}_l)/PSp_n(\mathbf{F}_l) \xrightarrow{\sim} (\mathbf{F}_l^\times)/(\mathbf{F}_l^\times)^2.$$

Thus $\overline{\mathbf{Q}}^{\ker \mathrm{ad}\,\overline{r}}$ does not contain $\mathbf{Q}(\zeta_l)$.

The corollary now follows from lemma 2.5.5 and theorem 4.4.3. $\square$

5. Ihara's lemma and the non-minimal case

The results of this chapter are not required for the proofs of the main theorems in [Tay] and [HSBT]. It could be skipped by those only interested in these applications.

5.1. *GL_n over a local field: finite characteristic theory II.* — We will keep the notation and assumptions of section 3.2. Following Vigneras we also make the following definition.

Definition 5.1.1 *We will call l* quasi-banal *for $GL_n(F_w)$ if either we have $l \not| \# GL_n(k(w))$ (the* banal case*), or we have $l > n$ and $q_w \equiv 1 \bmod l$ (the* limit case*).*

Suppose that U is an open subgroup of $GL_n(\mathcal{O}_{F_w})$ and that

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a $\overline{k}$-algebra homomorphism. Set

$$\begin{aligned}&\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \\ &= \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \otimes_{\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})],\phi} \overline{k}\end{aligned}$$

and

$$\begin{aligned}&\overline{k}[U\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \\ &= \overline{k}[U\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \otimes_{\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})],\phi} \overline{k}\end{aligned}$$

If V is any smooth $\overline{k}[GL_n(F_w)]$-module and if $v \in V^{GL_n(\mathcal{O}_{F_w})}$ satisfies $Tv = \phi(T)v$ for all $T \in \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$, then there is a unique map of $\overline{k}[GL_n(F_w)]$-modules

$$\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \longrightarrow V$$

sending $[GL_n(\mathcal{O}_{F_w})]$ to v, and a unique map of $\overline{k}[U\backslash GL_n(F_w)/U]$-modules

$$\overline{k}[U\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \longrightarrow V^U$$

sending $[GL_n(\mathcal{O}_{F_w})]$ to v. (These observations were previously used in a similar context by Lazarus [La].)

Fix an additive character $\psi : F_w \to \overline{k}$ with kernel $\mathcal{O}_{F_w}$. Let B_n denote the Borel subgroup of GL_n consisting of upper triangular matrices and let N_n denote its unipotent radical. Let P_n denote the subgroup of GL_n consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \in GL_{n-1}$. We will think of ψ as a character of $N_n(F_w)$ by

$$\psi : \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n-1} & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n-1} & a_{2n} \\ 0 & 0 & 1 & \dots & a_{3n-1} & a_{3n} \\ & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n-1n} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \longmapsto \psi(a_{12} + a_{23} + \dots + a_{n-1n}).$$

We will denote by gen_n the compact induction $\text{c-Ind}\,_{N_n(F_w)}^{P_n(F_w)} \psi$ and by $\mathcal{W}_n$ the induction $\mathrm{Ind}\,_{N_n(F_w)}^{GL_n(F_w)} \psi$. We will use the theory of derivatives over $\overline{k}$ as it is developed in section III.1 of [V1]. Note that if π is a smooth $\overline{k}[GL_n(F_w)]$-module then

$$\mathrm{Hom}\,_{GL_n(F_w)}(\pi, \mathcal{W}_n) \cong \pi^{\vee}_{N_n(F_w),\psi} \cong \mathrm{Hom}\,_{P_n(F_w)}(\mathrm{gen}_n, \pi)^{\vee},$$

where $\vee$ denote linear dual and $\pi_{N_n(F_w),\psi}$ denotes the maximal quotient of π on which $N_n(F_w)$ acts by ψ. If π is irreducible we will call it *generic* if these spaces are non-trivial.

The next lemma is proved exactly as in characteristic zero (see [Sh]).

Lemma 5.1.2 *Suppose that $\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \to \overline{k}$ is a homomorphism. Then the ϕ eigenspace in $\mathcal{W}_n^{GL_n(\mathcal{O}_{F_w})}$ is one dimensional and spanned by a function W_ϕ^0 with $W_\phi^0(1) = 1$.*

The next lemma is due to Vignéras, see parts 1 and 3 of theorem 1 of her appendix to this article.

Lemma 5.1.3 (Vignéras) *Suppose that l is quasi-banal for $GL_n(F_w)$. Then the functor $V \mapsto V^{\mathrm{Iw}(w)}$ is an equivalence of categories from the category of smooth $\overline{k}[GL_n(F_w)]$-modules generated by their* Iw(w)*-fixed vectors to the category of $\overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/\mathrm{Iw}(w)]$-modules. Moreover the category of smooth $\overline{k}[GL_n(F_w)]$-modules generated by their* Iw(w)*-fixed vectors is closed under passage to subquotients (in the category of smooth $\overline{k}[GL_n(F_w)]$-modules).*

Lemma 5.1.4 *Suppose that l is quasi-banal for $GL_n(F_w)$ and that*

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a $\overline{k}$-algebra homomorphism. Then $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ has finite length (as a smooth $\overline{k}[GL_n(F_w)]$-module) and its Jordan-Holder constituents are the same as those of any unramified principal series representation π for which $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ acts on $\pi^{GL_n(\mathcal{O}_{F_w})}$ by ϕ. In particular the smooth representation $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ has exactly one generic irreducible subquotient.

Proof: In the banal case this is due to Lazarus [La]. By lemma 5.1.3 the $\mathrm{Iw}(w)$-invariants functor is exact on the category of subquotients of smooth $\overline{k}[GL_n(F_w)]$-modules generated by their $\mathrm{Iw}(w)$-fixed vectors. The $\overline{k}[GL_n(F_w)]$-module $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ is generated by its $\mathrm{Iw}(w)$-fixed vectors.

Let the elements $T_1, ..., T_{n+1}$ generate $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ as a $\overline{k}$-algebra. Then we have a right exact sequence

$$\begin{array}{r} \bigoplus_{i=1}^{n+1} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \stackrel{\sum_i (T_i - \phi(T_i))}{\longrightarrow} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \\ \longrightarrow \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \longrightarrow (0). \end{array}$$

Taking $\mathrm{Iw}(w)$-invariants, we get an exact sequence

$$\begin{array}{l} \bigoplus_{i=1}^{n+1} \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \stackrel{\sum_i (T_i - \phi(T_i))}{\longrightarrow} \\ \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi^{\mathrm{Iw}(w)} \longrightarrow (0). \end{array}$$

We deduce that

$$(\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi)^{\mathrm{Iw}(w)} = \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi.$$

(We thank a referee for pointing out that the original argument we gave for this was needlessly complex.)

Following Kato and Lazarus [La] we see that the Satake isomorphism extends to an isomorphism

$$\overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \cong \overline{k}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$$

as $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \cong \overline{k}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^{S_n}$-modules. We deduce immediately that

$$\dim_{\overline{k}} \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi = n!$$

and hence (from lemma 5.1.3) that $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ has finite length. Moreover the argument of section 3.3 of [La] then shows that the Jordan-Holder constituents of $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ are the same as the Jordan-Holder constituents of any unramified principal series representation π for which $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ acts on $\pi^{GL_n(\mathcal{O}_{F_w})}$ by ϕ. The final assertion of the lemma then follows from the results of section III.1 of [V1]. $\square$

We will now recall some results of Russ Mann [Man1] and [Man2]. See also appendix A of this article.

The first result follows at once from proposition 4.4 of [Man1].

Lemma 5.1.5 (Mann) *Suppose that* $\chi_1, ..., \chi_n$ *are unramified characters* $F_w^\times \to \overline{K}^\times$ *and set* $\pi = \text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)$. *The simultaneous eigenspaces of the operators* $U_w^{(j)}$ *(for* $j = 1, ..., n-1$*) on* $\pi^{U_1(w^n)}$ *are parametrised by subsets* $A \subset \{1, ..., n\}$ *of cardinality less than* n. *Let* $u_A^{(j)}$ *denote the eigenvalue of* $U_w^{(j)}$ *on the eigenspace corresponding to* A. *Then*

$$\begin{aligned} &X^n - q_w^{(1-n)/2} u_A^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-n)/2} u_A^{(j)} X^{n-j} + \\ &+ ... + (-1)^{n-1} q_w^{(n-1)/2} u_A^{(n-1)} X = X^{n-\#A} \prod_{i \in A} (X - \chi_i(\varpi_w)). \end{aligned}$$

Moreover the generalised eigenspace corresponding to a subset A *has dimension* $\begin{pmatrix} n-1 \\ \#A \end{pmatrix}$.

The next two results are proved in [Man2]. As this is not currently available, the proofs are repeated in appendix A.

Lemma 5.1.6 (Mann) *Suppose that*

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a homomorphism. Then the map

$$\begin{array}{rcl} \overline{k}[U_1(w^n) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi & \longrightarrow & \mathcal{W}_n \\ T & \longmapsto & TW_\phi^0 \end{array}$$

is an injection.

Let η_w denote the diagonal matrix $\mathrm{diag}(1, ..., 1, \varpi_w^n)$. Then there is a bijection $\widehat{\ }$:

$$\begin{array}{rcl} \mathbf{Z}[1/q_w][U_1(w^n) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] & \to & \mathbf{Z}[1/q_w][GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w)/U_1(w^n)] \\ {[U_1(w^n) g GL_n(\mathcal{O}_{F_w})]} & \mapsto & [GL_n(\mathcal{O}_{F_w})\, {}^t g \eta_w^{-1} U_1(w^n)]. \end{array}$$

(This is because $U_1(w^n) = \eta_w {}^t U_1(w^n) \eta_w^{-1}$.)

Proposition 5.1.7 (Mann) *There exists an element*

$$\theta_{n,w} \in \mathbf{Z}_l[U_1(w^n) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F,w})]$$

with the following properties.

1. *For* $i = 1, ..., n-1$ *we have* $U_w^{(i)} \theta_{n,w} = 0$.
2. *For any homomorphism* $\phi : \overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \to \overline{k}$ *we have* $\theta_{n,w} W_\phi^0 \neq 0$ *in* $\mathcal{W}_n$.

3. If $\chi_1, ..., \chi_n$ are unramified characters $F_w^\times \rightarrow K^\times$ such that the induced representation $\pi = \text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)$ is irreducible and if v is a nonzero element of $\pi^{GL_n(\mathcal{O}_{F_w})}$, then $\theta_{n,w}v$ is nonzero and so a basis of $\pi^{U_1(w^n), U_w^{(1)}=...=U_w^{(n-1)}=0}$.

4. The composite

$$\widehat{\theta}_{n,w}\theta_{n,w} \in \mathbf{Z}_l[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

has Satake transform

$$q_w^{n^2(n-1)/2}(X_1...X_n)^{-(n+1)}\prod_{i=1}^{n}\prod_{j=1}^{n}(q_wX_i - X_j).$$

Corollary 5.1.8 *Suppose that π is an irreducible unramified representation of $GL_n(F_w)$ over $\overline{K}$ such that $r_l(\pi)^\vee(1-n)$ is defined over K. If $\widehat{\theta}_{n,w}\theta_{n,w}$ acts on $\pi^{GL_n(\mathcal{O}_{F,w})}$ by α then $\alpha \in \mathcal{O}$ and*

$$\mathrm{lg}_{\mathcal{O}}\, \mathcal{O}/\alpha \geq \mathrm{lg}_{\mathcal{O}}\, H^0(\mathrm{Gal}\,(\overline{F}_w/F_w), (\mathrm{ad}\, r_l(\pi)^\vee(1-n)) \otimes_{\mathcal{O}} (K/\mathcal{O})(-1)).$$

Definition 5.1.9 *Let M be an admissible $\overline{k}[GL_n(F_w)]$-module. We will say that M has the* Ihara property *if for every $v \in M^{GL_n(\mathcal{O}_{F_w})}$ which is an eigenvector of $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$, every irreducible submodule of the $\overline{k}[GL_n(F_w)]$-module generated by v is generic.*

Lemma 5.1.10 *Suppose that l is quasi-banal for $GL_n(F_w)$. Suppose also that M is an admissible $\overline{k}[GL_n(F_w)]$-module with the Ihara property and that*

$$\ker(\theta_{n,w} : M^{GL_n(\mathcal{O}_{F_w})} \longrightarrow M)$$

is a $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$-module. Then

$$\theta_{n,w} : M^{GL_n(\mathcal{O}_{F_w})} \hookrightarrow M^{U_1(w^n), U_w^{(1)}=...=U_w^{(n-1)}=0}$$

is injective.

Proof: Suppose $\theta_{n,w}$ were not injective on $M^{GL_n(\mathcal{O}_{F,w})}$. We could choose a $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$-eigenvector $0 \neq v \in \ker\theta_{n,w}$, say

$$Tv = \phi(T)v$$

where

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a $\overline{k}$-algebra homomorphism.

Let A denote the kernel of the map

$$\begin{aligned} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi &\longrightarrow \mathcal{W}_n \\ T &\longmapsto TW^0_\phi. \end{aligned}$$

Thus A has no generic subquotient and $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi/A$ has a unique irreducible submodule B/A. The module B/A is generic, but no subquotient of $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi/B$ is generic.

Now consider the map

$$\begin{aligned} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi &\longrightarrow M \\ T &\longmapsto Tv. \end{aligned}$$

As M has the Ihara property, any irreducible submodule of the image is generic. Thus A is contained in the kernel and moreover the induced map

$$\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi/A \longrightarrow M$$

must be injective. Thus we have an injection

$$\begin{aligned} \langle GL_n(F_w)W^0_\phi\rangle &\hookrightarrow M \\ W^0_\phi &\longmapsto v. \end{aligned}$$

Proposition 5.1.7 then tells us that $\theta_{n,w}v \neq 0$, a contradiction. $\square$

We would conjecture that the previous lemma remains true without the quasi-banal hypothesis. In fact, it is tempting to conjecture that the natural map

$$\begin{aligned} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi &\longrightarrow \mathcal{W}_n \\ [GL_n(\mathcal{O}_{F_w})] &\longmapsto W^0_\phi \end{aligned}$$

is in general injective.

5.2. *Duality.* — Keep the notation of section 3.3. In this section we will develop a duality theory for automorphic forms on G. It will actually pair automorphic forms on G with automorphic forms on another related group G'. So first we define an algebraic group G'/F^+ by setting

$$G'(R) = \{g \in B^{\mathrm{op}} \otimes_{F^+} R : \ g^{\ddagger\otimes 1}g = 1\}$$

for any F^+-algebra R. Note that there is an isomorphism

$$\begin{aligned} I : G &\xrightarrow{\sim} G' \\ g &\longmapsto g^{-1}. \end{aligned}$$

Our choice of an order $\mathcal{O}_B$ in B gives a model of G' over $\mathcal{O}_{F^+}$. If $v = ww^c$ splits in F then $i_v^t : \mathcal{O}_{B,v}^{\mathrm{op}} \xrightarrow{\sim} M_n(\mathcal{O}_{F_v})$ and we get an identification

$$\begin{aligned} i_w^t : G'(F_v^+) &\xrightarrow{\sim} GL_n(F_w) \\ (i_v^t)^{-1}(x, {}^t x^{-c}) &\longmapsto x \end{aligned}$$

with $i_w^t G'(\mathcal{O}_{F^+,v}) = GL_n(\mathcal{O}_{F,w})$ and $i_w^t \circ I = {}^t(i_w)^{-1} = c \circ i_{w^c}$. If $v \in S(B)$ and w is a prime of F above v we get an isomorphism

$$i_w' : G'(F_v^+) \xrightarrow{\sim} (B_w^{\mathrm{op}})^\times$$

with $i_w' G'(\mathcal{O}_{F^+,v}) = \mathcal{O}_{B^{\mathrm{op}},w}^\times$.

Given an n-tuple of integers $a = (a_1, ..., a_n)$ with $a_1 \geq ... \geq a_n$ there is a (unique up to scalar multiples) perfect pairing

$$\langle\ ,\ \rangle_a : W_a \times W_a \longrightarrow \mathbf{Q}$$

such that

$$\langle \xi_a(g)w, w' \rangle_a = \langle w, \xi_a({}^t g)w' \rangle_a$$

for all $w, w' \in W_a$ and $g \in GL_n(\mathbf{Q})$. Let $M_a' \subset W_a$ denote the $\langle\ ,\ \rangle_a$ dual of M_a and

$$\xi_a' : GL_n \longrightarrow GL(M_a')$$

the corresponding model over $\mathbf{Z}$ of ξ_a.

If $a \in \mathrm{Wt}_n$ then there is an irreducible representation

$$\begin{aligned} \xi_a' : G'(F_l^+) &\longrightarrow GL(W_a) \\ g &\longmapsto \textstyle\prod_{\tau \in \widetilde{I}_l} \xi_{a_\tau}(\tau i_\tau^t g). \end{aligned}$$

The representation ξ_a' contains a $G'(\mathcal{O}_{F^+,l})$-invariant $\mathcal{O}$-lattice M_a' such that there is a perfect pairing

$$\langle\ ,\ \rangle_a : M_a \times M_a' \longrightarrow \mathcal{O}$$

with

$$\langle \xi_a(g)x, \xi_a'(I(g))y \rangle_a = \langle x, y \rangle_a.$$

For $v \in S(B)$, let $M_{\rho_v}' = \mathrm{Hom}\,(M_{\rho_v}, \mathcal{O})$ and define $\rho_v' : G(F_v^+) \to GL(M_{\rho_v}')$ by

$$\rho_v'(g)(x)(y) = x(\rho_v(I^{-1}(g))^{-1}y).$$

If we identify $G(F_v^+) \cong B_w^\times$ and $G'(F_v^+) \cong (B_w^{\mathrm{op}})^\times$ and if $g \in B_w^\times$ and $g' \in (B_w^{\mathrm{op}})^\times$ have the same characteristic polynomials then $\mathrm{tr}\,\rho_v(g) = \mathrm{tr}\,\rho_v(g')$. We have $\mathrm{JL}\,(\rho_v' \circ i_w^{-1}) = \mathrm{Sp}_{\,m_v}(\pi_w)$.

For $v \in R$ let $U_{0,v}'$ be an open compact subgroup of $G'(F_v^+)$ and let

$$\chi_v' : U_{0,v}' \longrightarrow \mathcal{O}^\times$$

be a homomorphism with open kernel.

Let A denote an $\mathcal{O}$-algebra. Suppose that $a \in \mathrm{Wt}_n$ and that for $v \in S(B)$, ρ_v is as in section 3.3. Set

$$M_{a,\{\rho_v\},\{\chi'_v\}} = M'_a \otimes \left(\bigotimes_{v \in S(B)} M'_{\rho_v} \right) \otimes \left(\bigotimes_{v \in R} \mathcal{O}(\chi'_v) \right).$$

If U' is an open compact subgroup of $G'(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v \in R} U'_{0,v}$ and either A is a K-algebra or the projection of U' to $G'(F_l^+)$ is contained in $G'(\mathcal{O}_{F^+,l})$ we define

$$S'_{a,\{\rho_v\},\{\chi'_v\}}(U', A)$$

to be the space of functions

$$f : G'(F^+) \backslash G'(\mathbf{A}_{F^+}^{\infty}) \longrightarrow A \otimes_{\mathcal{O}} M'_{a,\{\rho_v\},\{\chi'_v\}}$$

such that

$$f(gu) = u_{S_l \cup S(B) \cup R}^{-1} f(g)$$

for all $u \in U'$ and $g \in G'(\mathbf{A}_{F^+}^{\infty})$. As in section 3.3 we extend this to define $S'_{a,\{\rho_v\},\{\chi'_v\}}(V', A)$ for V' any compact subgroup of $G'(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v \in R} U'_{0,v}$ and define actions of $g' \in G'(\mathbf{A}_{F^+}^{R,\infty}) \times \prod_{v \in R} U'_{0,v}$ and of Hecke operators $[U'_1 g' U'_2]$.

Lemma 3.3.1, proposition 3.3.2, corollary 3.3.3 and proposition 3.3.4 all remain true for G'.

Suppose that U is an open compact subgroup of $G(\mathbf{A}_{F^+}^{R,\infty} \times \prod_{v \in R} U_{0,v})$ and that $\eta \in G'(\mathbf{A}_{F^+}^{\infty})$. Suppose also that for v an element of R we have $U'_{v,0} = \eta_v^{-1} I(U_{v,0}) \eta_v$ and

$$\chi'_v(u'_v) = (\chi_v \circ I^{-1})(\eta_v u'_v \eta_v^{-1})^{-1}.$$

If A is not a K-algebra further assume that $\eta_l \in G'(\mathcal{O}_{F^+,l})$ and that for all $u \in U$ we also have $u_l \in G(\mathcal{O}_{F^+,l})$. Set $U' = \eta^{-1} I(U) \eta$. Define a pairing

$$\langle \ , \ \rangle_{U,\eta} : S_{a,\{\rho_v\},\{\chi_v\}}(U, A) \times S'_{a,\{\rho_v\},\{\chi'_v\}}(U', A) \longrightarrow A$$

by

$$\langle f, f' \rangle_{U,\eta} = \sum_{g \in G(F^+) \backslash G(\mathbf{A}_{F^+}^{\infty})/U} \langle f(g), \eta_{S_l \cup S(B)} f'(I(g)\eta) \rangle_{a,\{\rho_v\}}.$$

If U is sufficiently small, or if A is a K-algebra, this is a perfect pairing. If we have two such pairs (U_1, η_1) and (U_2, η_2) with each U_i sufficiently small, if $U'_i = \eta_i^{-1} I(U_i) \eta_i$ and if $g \in G(\mathbf{A}_{F^+}^{\infty})$ (with $g_l \in G(\mathcal{O}_{F^+,l})$ if A is not a K-algebra) then

$$\langle [U_1 g U_2] f, f' \rangle_{U_1,\eta_1} = \langle f, [U'_2 \eta_2^{-1} I(g)^{-1} \eta_1 U'_1] f' \rangle_{U_2,\eta_2}.$$

Now suppose that

$$U' = \prod_v U'_v \subset G'(\mathbf{A}^\infty_{F^+})$$

is a sufficiently small open compact subgroup , that $T \supset S(B) \cup R$ and that, if $v \notin T$ splits in F, then $U'_v = G(\mathcal{O}_{F^+,v})$. We will denote by

$$\mathbf{T}^T_{a,\{\rho_v\},\{\chi'_v\}}(U')'$$

the $\mathcal{O}$-subalgebra of $\mathrm{End}\,(S'_{a,\{\rho_v\},\{\chi'_v\}}(U',\mathcal{O}))$ generated by the Hecke operators $T^{(j)}_w$ (or strictly speaking $(i^t_w)^{-1}(T^{(j)}_w) \times (U')^v$) and $(T^{(n)}_w)^{-1}$ for $j = 1, ..., n$ and for w a place of F which is split over a place $v \notin T$ of F^+. (Again $T^{(j)}_{w^c} = (T^{(n)}_w)^{-1} T^{(n-j)}_w$, so we need only consider one place w above a given place v of F^+.) If X' is a $\mathbf{T}^T_{a,\{\rho_v\}}(U')'$-stable subspace of $S'_{a,\{\rho_v\},\{\chi'_v\}}(U',K)$ then we will write

$$\mathbf{T}^T(X')'$$

for the image of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi'_v\}}(U')'$ in $\mathrm{End}\,_K(X')$. Note that $\mathbf{T}^T(X')'$ are finite and free as $\mathcal{O}$-modules and is reduced.

Proposition 3.4.2 remains true for G'. We call a maximal ideal $\mathfrak{m}'$ of $\mathbf{T}^T_{a,\{\rho_v\},\{\chi'_v\}}(U')'$ *Eisenstein* if $\bar{r}_{\mathfrak{m}'}$ is absolutely reducible. Then proposition 3.4.4, corollary 3.4.5 and lemma 3.4.1 also remain true for G'.

5.3. *Ihara's lemma and raising the level.* — Keep the notation and assumptions of sections 3.4 and 5.2.

In this section we will discuss congruences between modular forms of different levels. Unfortunately we can not prove anything. Rather we will explain how the congruence results we expect would follow from an analogue of Ihara's lemma for elliptic modular forms (see [I], [Ri]). Let us first describe this conjecture more precisely.

Conjecture I *Let G, l, T and U be as in section 3.4 with U sufficiently small. Suppose that $v \in T - (S(B) \cup S_l)$ with $U_v = G(\mathcal{O}_{F^+,v})$ and that $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbf{T}^T_{0,\{1\},\{1\}}(U)$. If $f \in S_{0,\{1\},\{1\}}(U, \bar{k})[\mathfrak{m}]$ and if π is an irreducible $G(F^+_v)$-submodule of*

$$\langle G(F^+_v) f\rangle \subset S_{0,\{1\},\{1\}}(U^v, \bar{k})$$

then π is generic.

In fact we suspect something stronger is true. Although we will not need this stronger form we state it here. We will call an irreducible $G(F^+_v)$-submodule π of $S_{a,\{\rho_x\},\{\chi_x\}}(\{1\}, \bar{k})$ *Eisenstein* if for some (and hence all) open compact subgroups $U = \prod_x U_x$ with $\pi^U \neq (0)$ there is a finite set $T' \supset R \cup S_l \cup S(B) \cup \{v\}$ of split primes and an Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbf{T}^{T'}_{a,\{\rho_x\},\{\chi_x\}}(U,\overline{k})$ with $\pi_{\mathfrak{m}} \neq (0)$.

Conjecture II *Let G and l be as in section 3.3. Suppose that $v \notin S(B) \cup S_l \cup R$ is a prime of F^+ which splits in F. Let π be a non-Eisenstein irreducible $G(F_v^+)$-submodule of $S_{0,\{1\},\{1\}}(\{1\},\overline{k})$. Then π is generic.*

We should point out that these conjectures are certainly false if we replace 'submodule' by 'subquotient'. If we replace $\overline{k}$ by $\overline{K}$ and $\mathbf{T}^{T'}_{0,\{1\},\{1\}}(U)$ by $\mathbf{T}^{T'}_{0,\{1\},\{1\}}(U) \otimes_{\mathcal{O}} K$, then the conjectures would be true by part 7 of proposition 3.3.4. In the case $n = 2$ the conjecture is an easy consequence of the strong approximation theorem for G. We also believe that we can prove many cases of conjecture I in the case $n = 3$. We hope to return to the case $n = 3$ in another paper.

Lemma 5.3.1 *Conjecture II (and hence conjecture I) is true if $n = 2$.*

Proof: Let G_1 denote the derived subgroup of G. Then we have exact sequences

$$(0) \longrightarrow G_1(F^+) \longrightarrow G(F^+) \stackrel{\det}{\longrightarrow} F^{\mathbf{N}_{F/F^+}=1}$$

and

$$(0) \longrightarrow G_1(\mathbf{A}^{\infty}_{F^+}) \longrightarrow G(\mathbf{A}^{\infty}_{F^+}) \stackrel{\det}{\longrightarrow} \mathbf{A}_F^{\mathbf{N}_{F/F^+}=1}.$$

Suppose π is as in the statement of conjecture II, but π is not generic. Then π is one dimensional and trivial on $G_1(F_v^+)$. Let $0 \neq f \in \pi$ be invariant by an open compact U. Then for all $g \in G(\mathbf{A}^{\infty}_{F^+})$, the function f is constant on

$$G(F^+)gUG_1(F_v^+) = G(F^+)G_1(\mathbf{A}^{\infty}_{F^+})gU$$

(by the strong approximation theorem). Thus f factors through

$$\det : G(F^+)\backslash G(\mathbf{A}^{\infty}_{F^+})/U \longrightarrow \det G(F^+)\backslash (\mathbf{A}_F^{\infty})^{\mathbf{N}=1}/\det U.$$

Thus we can find a character

$$\chi : \det G(F^+)\backslash (\mathbf{A}_F^{\infty})^{\mathbf{N}=1}/\det U \longrightarrow \overline{k}^{\times}$$

such that

$$\sum_{g \in (\det G(F^+))\backslash(\det G(\mathbf{A}^{\infty}_{F^+}))/(\det U)} \chi(g)^{-1} f(g) \neq 0.$$

It follows that, for all but finitely many places w of F which are split over F^+, $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_w)$ has characteristic polynomial

$$(X - \chi(\varpi_w/\varpi_w^c))(X - q_w \chi(\varpi_w/\varpi_w^c)).$$

We deduce that

$$(\mathrm{ad}\, \overline{r}_{\mathfrak{m}})^{\mathrm{ss}} = \epsilon^{-1} \oplus 1 \oplus 1 \oplus \epsilon.$$

Thus $\overline{r}_{\mathfrak{m}}$ is reducible and $\mathfrak{m}$ is Eisenstein. $\Box$

Lemma 5.3.2 *Let G be as in the section 3.4. Suppose conjecture I holds for all T and U with U sufficiently small. Let T, U, a, $\{\rho_x\}$ and $\{\chi_x\}$ be as in section 3.4. Let $v \in T - (S(B) \cup S_l \cup R)$ with $U_v = G(\mathcal{O}_{F^+,v})$ and let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}^T_{a,\{\rho_x\},\{\chi_x\}}(U)$. If $f \in S_{a,\{\rho_x\},\{\chi_x\}}(U, \overline{k})[\mathfrak{m}]$ and if π is an irreducible $G(F_v^+)$-submodule of*

$$\langle G(F_v^+) f \rangle \subset S_{a,\{\rho_x\},\{\chi_x\}}(U^v, \overline{k})$$

then π is generic.

Proof: We need only prove the lemma for U small, because its truth for some U implies its truth for all $U' \supset U$. But for U small enough we have

$$S_{a,\{\rho_x\},\{\chi_x\}}(U, \overline{k}) = S_{0,\{1\},\{1\}}(U, \overline{k})^r$$

for some r. $\Box$

We now turn to the construction of 'raising the level' congruences. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}^T_{a,\{\rho_x\},\{\chi_x\}}(U)$ and let

$$\phi : \mathbf{T}^T_{a,\{\rho_x\},\{\chi_x\}}(U)_{\mathfrak{m}} \longrightarrow \mathcal{O}.$$

We will consider subsets $S \subset T - (S(B) \cup S_l \cup R)$ such that $U_v = G(\mathcal{O}_{F^+,v})$ for all $v \in S$. For such S set

$$U(S) = U^S \prod_{v \in S} i_{\widetilde{v}}^{-1} U_1(\widetilde{v}^n)$$

and

$$\theta_S = \prod_{v \in R} i_{\widetilde{v}}^{-1} \theta_{n,\widetilde{v}}$$

and

$$X_S = S_{a,\{\rho_x\},\{\chi_x\}}(U(S), \mathcal{O})_{\mathfrak{m},\mathfrak{n}}$$

where $\mathfrak{n}$ denotes the maximal ideal

$$(\lambda, U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)} :\ v \in S)$$

of $\mathcal{O}[U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)} :\ v \in S]$. Further set

$$\mathbf{T}_S = \mathbf{T}^T(X_S),$$

so that $\mathbf{T}_\emptyset = \mathbf{T}^T_{a,\{\rho_x\},\{\chi_x\}}(U)_\mathfrak{m}$. If $S_1 \subset S_2$ are such sets then we get an injection

$$\theta_{S_2-S_1} : X_{S_1} \hookrightarrow X_{S_2}.$$

(To see that this map is an injection we may suppose that $S_2 = S_1 \cup \{v\}$. Let π be an irreducible constituent of $S_{a,\{\rho_x\},\chi_x\}}(\{1\}, \overline{K})$ with $\pi \cap X_{S_1} \neq (0)$. Because $\mathfrak{m}$ is not Eisenstein we see that π_v is generic (see part 7 of proposition 3.3.4). Thus by proposition 5.1.7

$$i_{\widetilde{v}}^{-1}\theta_{n,\widetilde{v}} : \pi \cap X_{S_1} \hookrightarrow \pi \cap X_{S_2}.)$$

Thus we also have a surjection

$$\mathbf{T}_{S_2} \twoheadrightarrow \mathbf{T}_{S_1}$$

which takes $T_w^{(j)}$ to $T_w^{(j)}$ for all w (a prime of F which is split over a prime of F^+ not in T) and j $(= 1, ..., n)$. Let ϕ_S denote the composite

$$\phi_S : \mathbf{T}_S \twoheadrightarrow \mathbf{T}_\emptyset \stackrel{\phi}{\longrightarrow} \mathcal{O}.$$

We will be interested in congruences between ϕ and other homomorphisms $\mathbf{T}_S \to \overline{K}$. In particular we will be interested in how these congruences vary with S. A useful measure of these congruences is provided by the ideal $\mathfrak{c}_S(\phi)$, defined by

$$\phi_S : \mathbf{T}_S/(\ker \phi_S + \mathrm{Ann}_{\mathbf{T}_S} \ker \phi_S) \stackrel{\sim}{\longrightarrow} \mathcal{O}/\mathfrak{c}_S(\phi).$$

Let $X_S[\phi]$ denote the subspace of X_S where $\mathbf{T}_S$ acts via ϕ_S. Let $i_S : X_S[\phi] \hookrightarrow X_S$ denote the canonical inclusion and let $\pi_S : X_S \twoheadrightarrow X_S[\phi]$ denote the $\mathbf{T}_S$-equivariant projection. (This exists because $\mathbf{T}_S$ is reduced.) The next lemma is now clear.

Lemma 5.3.3 *Keep the above notation. The module $X_S[\phi]/\pi_S i_S X_S[\phi]$ is an $\mathcal{O}/\mathfrak{c}_S(\phi)$-module. If X_S is free over $\mathbf{T}_S$ then $X_S[\phi]/\pi_S i_S X_S[\phi]$ is free over $\mathcal{O}/\mathfrak{c}_S(\phi)$.*

Lemma 5.3.4 *Keep the above notation. Then*

$$\theta_S : X_\emptyset[\phi] \otimes_\mathcal{O} K \stackrel{\sim}{\longrightarrow} X_S[\phi] \otimes_\mathcal{O} K.$$

Proof: It suffices to prove that if π is an irreducible constituent of the space $S_{a,\{\rho_x\},\{\chi_x\}}(\{1\}, \overline{K})$ then

$$\theta_S : (X_\emptyset[\phi] \otimes_\mathcal{O} \overline{K}) \cap \pi \stackrel{\sim}{\longrightarrow} (X_S[\phi] \otimes_\mathcal{O} \overline{K}) \cap \pi.$$

As $\phi r_\mathfrak{m}$ is unramified at $v \in S$, proposition 3.3.4 tells us that if $(X_S[\phi] \otimes_\mathcal{O} \overline{K}) \cap \pi \neq (0)$ then π_v is unramified. In particular $(X_\emptyset[\phi] \otimes_\mathcal{O} \overline{K}) \cap \pi \neq (0)$. If

$(X_\emptyset[\phi] \otimes_{\mathcal{O}} \overline{K}) \cap \pi \neq (0)$ then for $v \in S$ the representation π_v is unramified and, by part 7 of proposition 3.3.4, generic. Write

$$\pi_v \circ i_{\widetilde{v}}^{-1} = \text{n-Ind}\,_{B_n(F_{\widetilde{v}})}^{GL_n(F_{\widetilde{v}})}(\chi_{v,1}, ..., \chi_{v,n})$$

with each $\chi_{v,i}$ unramified. Again by proposition 3.3.4 we see that for $v \in S$, each $\chi_{v,i}(\varpi_{\widetilde{v}}) \in \mathcal{O}_{\overline{K}}^\times$. From lemma 5.1.5 we deduce that

$$\pi_{\mathfrak{n}}^{U(S)}$$

is the subspace of $\pi^{U(S)}$ on which $i_{\widetilde{v}}^{-1} U_{\widetilde{v}}^{(j)} = 0$ for each $v \in S$ and each $j = 1, ..., n-1$. Proposition 5.1.7 then tells us that

$$\theta_S : \pi^{U(\emptyset)} \xrightarrow{\sim} \pi_{\mathfrak{n}}^{U(S)}$$

as desired. □

Proposition 5.3.5 *Keep the above notation and assumptions. In particular assume that U is sufficiently small. Let $S \subset T - (S(B) \cup S_l \cup R)$ be such that $U_x = G(\mathcal{O}_{F^+,x})$ and $G'(\mathcal{O}_{F^+,x})$ for all $x \in S$. Suppose that conjecture I is true for the groups G and G', for l, for T, for $v \in S$, and for the various open compact subgroups U_{S_1} with $S_1 \subset S - \{v\}$. Also suppose that $X_\emptyset$ is free over $\mathbf{T}_\emptyset$. Finally suppose that for each $v \in S$, l is quasi-banal for $G(F_v^+)$. Then*

$$\mathrm{lg}_{\mathcal{O}}\, \mathcal{O}/\mathfrak{c}_S(\phi) \geq \mathrm{lg}_{\mathcal{O}}\, \mathcal{O}/\mathfrak{c}_\emptyset(\phi) + \sum_{v \in S} \mathrm{lg}_{\mathcal{O}}\, H^0(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), (\mathrm{ad}\, r_{\mathfrak{m}}) \otimes_{\mathbf{T}_\emptyset, \phi} K/\mathcal{O}(\epsilon^{-1})).$$

Proof: Let $\eta_\emptyset \in G'(\mathbf{A}_{F^+}^\infty)$ equal 1 at all places in $(S \cup S(B) \cup S_l)$ and all places outside T. If $S_1 \subset S$ set

$$\eta_{S_1} = \eta_\emptyset \prod_{v \in S_1} (i_{\widetilde{v}}^t)^{-1} \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \varpi_{\widetilde{v}}^n \end{pmatrix}$$

and

$$U(S_1)' = \eta_{S_1}^{-1} U(S_1) \eta_{S_1} = (U(\emptyset)')^{S_1} \times \prod_{v \in S_1} (i_{\widetilde{v}}^t)^{-1} U_1(\widetilde{v}^n).$$

Let $\mathfrak{m}'$ denote the ideal of $\mathbf{T}_{a,\{\rho_v\}}^T(U(S_1)')'$ generated by λ and $T_w^{(j)} - a$ whenever $a \in \mathcal{O}$, w is a prime of F split above a prime of F^+ not in T and $T_w^{(j)} - a \in \mathfrak{m}$. Then $\mathfrak{m}'$ is either maximal or the whole Hecke algebra. Set

$$X'_{S_1} = S'_{a,\{\rho_x\},\{\chi_x\}}(U(S_1)', \mathcal{O})_{\mathfrak{m}',\mathfrak{n}}$$

where $\mathfrak{n}$ denotes the maximal ideal

$$(\lambda, U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)})$$

of $\mathcal{O}[U_{\widetilde{v}}^{(1)},...,U_{\widetilde{v}}^{(n-1)}]$, and
$$\mathbf{T}'_{S_1}=\mathbf{T}^T(X_{S_1})'.$$
Also set
$$\theta'_{S_1}=\prod_{v\in S_1}(i^t_{\widetilde{v}})^{-1}\theta_{n,\widetilde{v}}$$
and
$$\widehat{\theta}'_{S_1}=\prod_{v\in S_1}(i^t_{\widetilde{v}})^{-1}(\widehat{\theta}_{n,\widetilde{v}}).$$
If $S_1\subset S_2\subset S$ then we get an injection
$$\theta'_{S_2-S_1}:X'_{S_1}\hookrightarrow X'_{S_2}$$
and exactly as in the proof of lemma 5.3.4 we see that
$$\theta'_{S_2-S_1}:X'_{S_1}\otimes_{\mathcal{O}}K\xrightarrow{\sim}X'_{S_2}\otimes_{\mathcal{O}}K.$$
Also by corollary 3.4.5
$$\widehat{\theta}'_{S_1}\theta_{S_1}=\prod_{v\in S_1}i_{\widetilde{v}}^{-1}(\widehat{\theta}_{n,\widetilde{v}}\theta_{n,\widetilde{v}})$$
acts on $X_\emptyset$ by an element of $\mathbf{T}_\emptyset$.

Under the perfect pairing
$$\langle\ ,\ \rangle_{U(S_1),\eta_{S_1}}:S_{a,\{\rho_x\},\{\chi_x\}}(U(S_1),\mathcal{O})\times S'_{a,\{\rho_x\},\{\chi_x\}}(U(S_1)',\mathcal{O})\longrightarrow\mathcal{O}$$
we have that:

- for $v\in S_1$ the adjoint of $i_{\widetilde{v}}^{-1}U_{\widetilde{v}}^{(j)}$ is $(i^t_{\widetilde{v}})^{-1}U_{\widetilde{v}}^{(j)}$, and
- for w a prime of F split over a prime of F^+ not in T, the adjoint of $T_w^{(j)}$ is $T_w^{(j)}$.

Thus $\mathbf{T}_{S_1}\cong\mathbf{T}'_{S_1}$ (with $T_w^{(j)}$ matching $T_w^{(j)}$ for w a prime of F split over a prime of F^+ not in T), and $\langle\ ,\ \rangle_{U(S_1),\eta_{S_1}}$ induces a perfect pairing
$$\langle\ ,\ \rangle_{S_1}:X_{S_1}\times X'_{S_1}\longrightarrow\mathcal{O}$$
under which the actions of $\mathbf{T}_{S_1}\cong\mathbf{T}'_{S_1}$ are self-adjoint. If $S_1\subset S_2\subset S$, then
$$\widehat{\theta}'_{S_2-S_1}:X_{S_2}\longrightarrow X_{S_1}$$
is the adjoint of $\theta'_{S_2-S_1}$.

It follows from conjecture I and lemma 5.1.10 that
$$\theta_{\{v\}}:X_{S_1}\longrightarrow X_{S_1\cup\{v\}}$$

has torsion free cokernel, and that

$$\widehat{\theta}'_{\{v\}} : X_{S_1 \cup \{v\}} \longrightarrow X_{S_1}$$

is surjective. Thus

$$\theta_S : X_\emptyset \longrightarrow X_S$$

has torsion free cokernel, and

$$\widehat{\theta}'_S : X_S \longrightarrow X_\emptyset$$

is surjective. In particular

$$\theta_S : X_\emptyset[\phi] \xrightarrow{\sim} X_S[\phi],$$

and we may take

$$i_S = \theta_S \circ i_\emptyset \circ \theta_S|_{X_\emptyset[\phi]}^{-1}$$

and

$$\pi_S = \theta_S|_{X_\emptyset[\phi]} \circ \pi_\emptyset \circ \widehat{\theta}'_S.$$

Thus

$$\begin{aligned} X_S[\phi]/\pi_S i_S X_S[\phi] &\cong X_\emptyset[\phi]/\phi(\widehat{\theta}'_S \theta_S)\pi_\emptyset i_\emptyset X_\emptyset[\phi] \\ &\cong X_\emptyset[\phi]/(\textstyle\prod_{v \in S} \phi i_{\tilde{v}}^{-1}(\widehat{\theta}_{n,\tilde{v}} \theta_{n,\tilde{v}}))\pi_\emptyset i_\emptyset X_\emptyset[\phi]. \end{aligned}$$

The proposition follows from corollary 5.1.8. □

5.4. $R = \mathbf{T}$ *theorems: the non-minimal case.* — In this section we will show how conjecture I would imply a generalisation of theorem 3.5.1 to a less restrictive set of deformation problems $\mathcal{S}$. Such a generalisation would be very much more useful in practice than theorem 3.5.1. After this paper was written, one of us (R.L.T.) found an unconditional proof of a slight weakening of theorem 5.4.1 below (see [Tay]). This seems to be sufficient for most current applications. However we present this conditional argument here because it would provide a stronger result. For instance it shows that the Galois deformation ring is a reduced complete intersection, which might be pertinent for special value conjectures. This information does not appear to be available by the methods of [Tay].

For the sake of clarity we recap the notation.

Fix a positive integer $n \geq 2$ and a prime $l > n$.

Fix an imaginary quadratic field E in which l splits and a totally real field F^+ such that

- $F = F^+E/F^+$ is unramified at all finite primes, and
- $F^+/\mathbf{Q}$ is unramified at l.

Fix a finite non-empty set of places $S(B)$ of places of F^+ with the following properties:

– Every element of $S(B)$ splits in F.
– $S(B)$ contains no place above l.
– If n is even then

$$n[F^+ : \mathbf{Q}]/2 + \#S(B) \equiv 0 \bmod 2.$$

Choose a division algebra B with centre F with the following properties:

– $\dim_F B = n^2$.
– $B^{\mathrm{op}} \cong B \otimes_{E,c} E$.
– B splits outside $S(B)$.
– If w is a prime of F above an element of $S(B)$, then B_w is a division algebra.

Fix an involution $\ddagger$ on B and define an algebraic group G/F^+ by

$$G'(A) = \{g \in B \otimes_{F^+} A : \; g^{\ddagger \otimes 1} g = 1\}$$

such that

– $\ddagger|_F = c$,
– for a place $v|\infty$ of F^+ we have $G(F_v^+) \cong U(n)$, and
– for a finite place $v \notin S(B)$ of F^+ the group $G(F_v^+)$ is quasi-split.

Also define an algebraic group G'/F^+ by setting

$$G'(A) = \{g \in B^{\mathrm{op}} \otimes_{F^+} A : \; g^{\ddagger \otimes 1} g = 1\}$$

for any F^+-algebra A.

Choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^{\ddagger} = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is maximal for all primes w of F which are split over F^+. This gives a model of G over $\mathcal{O}_{F^+}$. If $v \notin S(B)$ is a prime of F^+ which splits in F choose an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F,v})$ such that $i_v(x^{\ddagger}) = {}^t i_v(x)^c$. If w is a prime of F above v this gives rise to an isomorphism $i_w : G(F_v^+) \xrightarrow{\sim} GL_n(F_w)$ as in section 3.3. If $v \in S(B)$ and w is a prime of F above v choose isomorphisms $i_w : G(F_v^+) \xrightarrow{\sim} B_w^{\times}$ such that $i_{w^c} = i_w^{-\ddagger}$ and $i_w G(\mathcal{O}_{F^+,v}) = \mathcal{O}_{B,w}^{\times}$.

Let S_l denote the set of primes of F^+ above l. Let S_a denote a non-empty set, disjoint from $S_l \cup S(B)$, of primes of F^+ such that

– if $v \in S_a$ then v splits in F, and
– if $v \in S_a$ lies above a rational prime p then $[F(\zeta_p) : F] > n$.

Let S denote a set, disjoint from $S_l \cup S(B) \cup S_a$, of primes of F^+ such that

– if $v \in S$ then v splits in F, and

– if $v \in S$ then either $\mathbf{N}v \equiv 1 \bmod l$ or $l \not| \# GL_n(k(v))$.

Let $T = S \cup S(B) \cup S_l \cup S_a$. Let $\widetilde{T}$ denote a set of primes of F above T such that $\widetilde{T} \coprod \widetilde{T}^c$ is the set of all primes of F above T. If $v \in T$ we will let $\widetilde{v}$ denote the prime of $\widetilde{T}$ above v. If $T_1 \subset T$ we will let $\widetilde{T}_1$ denote the set of $\widetilde{v}$ for $v \in T_1$.

If $S_1 \subset S$ let $U(S_1) = \prod_v U(S_1)_v$ denote an open compact subgroup of $G(\mathbf{A}_{F^+}^\infty)$ such that

– if v is not split in F then $U(S_1)_v$ is a hyperspecial maximal compact subgroup of $G(F_v^+)$,
– if $v \notin S_a \cup S_1$ splits in F then $U(S_1)_v = G(\mathcal{O}_{F^+,v})$,
– if $v \in S_1$ then $U(S_1)_v = i_{\widetilde{v}}^{-1} U_1(\widetilde{v}^n)$, and
– if $v \in S_a$ then $U(S_1)_v = i_{\widetilde{v}}^{-1} \ker(GL_n(\mathcal{O}_{F,\widetilde{v}}) \to GL_n(\mathcal{O}_{F,\widetilde{v}}/(\varpi_{\widetilde{v}}^{m_v})))$ for some $m_v \geq 1$.

Then $U(S_1)$ is sufficiently small. If $S_1 = \emptyset$ we will drop it from the notation, i.e. we will write $U = \prod_v U_v$ for $U(\emptyset)$.

Let $K/\mathbf{Q}_l$ be a finite extension which contains the image of every embedding $F^+ \hookrightarrow \overline{K}$. Let $\mathcal{O}$ denote its ring of integers, λ the maximal ideal of $\mathcal{O}$ and k the residue field $\mathcal{O}/\lambda$.

For each $\tau : F \hookrightarrow K$ choose integers $a_{\tau,1}, ..., a_{\tau,n}$ such that

– $a_{\tau c,i} = -a_{\tau,n+1-i}$, and
– if τ gives rise to a place in $\widetilde{S}_l$ then

$$l - 1 - n \geq a_{\tau,1} \geq ... \geq a_{\tau,n} \geq 0.$$

For each $v \in S(B)$ let $\rho_v : G(F_v^+) \longrightarrow GL(M_{\rho_v})$ denote a representation of $G(F_v^+)$ on a finite free $\mathcal{O}$-module such that ρ_v has open kernel and $M_{\rho_v} \otimes_{\mathcal{O}} \overline{K}$ is irreducible. For $v \in S(B)$, define m_v, $\pi_{\widetilde{v}}$ and $\widetilde{r}_{\widetilde{v}}$ by

$$\mathrm{JL}\,(\rho_v \circ i_{\widetilde{v}}^{-1}) = \mathrm{Sp}_{\,m_v}(\pi_{\widetilde{v}})$$

and

$$\widetilde{r}_{\widetilde{v}} = r_l(\pi_{\widetilde{v}} |\ |^{(n/m_{\widetilde{v}}-1)(1-m_{\widetilde{v}})/2}).$$

We will suppose that

$$\widetilde{r}_{\widetilde{v}} : \mathrm{Gal}\,(\overline{F}_w/F_w) \longrightarrow GL_{n/m_{\widetilde{v}}}(\mathcal{O})$$

(as opposed to $GL_{n/m_{\widetilde{v}}}(\overline{K})$), that the reduction of $\widetilde{r}_{\widetilde{v}} \bmod \lambda$ is absolutely irreducible and that for $i = 1, ..., m_v$ we have

$$\widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k \not\cong \widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k(\epsilon^i).$$

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(U)$ with residue field k and let

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k)$$

be a continuous homomorphism associated to $\mathfrak{m}$ as in propositions 3.4.2 and 3.4.4. Note that

$$\nu \circ \overline{r}_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$$

where δ_{F/F^+} is the non-trivial character of $\mathrm{Gal}\,(F/F^+)$ and where $\mu_{\mathfrak{m}} \in \mathbf{Z}/2\mathbf{Z}$. We will *assume* that $\overline{r}_{\mathfrak{m}}$ has the following properties.

- $\overline{r}_{\mathfrak{m}}(\mathrm{Gal}\,(\overline{F}/F^+(\zeta_l)))$ is big in the sense of section 2.5.
- If $v \in S_a$ then $\overline{r}_{\mathfrak{m}}$ is unramified at v and

$$H^0(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), (\mathrm{ad}\,\overline{r}_{\mathfrak{m}})(1)) = (0).$$

We will also assume that $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(U)$ admits a section

$$\phi : \mathbf{T}^T_{a,\{\rho_v\},\emptyset}(U) \to \mathcal{O}.$$

For $S_1 \subset S$ write $X_{\mathfrak{m},S_1}$ for the space

$$S_{a,\{\rho_v\},\emptyset}(U(S_1), \mathcal{O})_{\mathfrak{m},\mathfrak{n}}$$

where $\mathfrak{n}$ is the maximal ideal

$$(\lambda, U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)} :\ v \in S_1)$$

of $\mathcal{O}[U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)} :\ v \in S_1]$. Also write $\mathbf{T}_{\mathfrak{m},S_1}$ for the algebra $\mathbf{T}^T(X_{\mathfrak{m},S_1})$. Thus $\mathbf{T}_{\mathfrak{m},S_1}$ is a quotient of $\mathbf{T}^T_{a,\{\rho_v\},\emptyset}(U(S_1))_{\mathfrak{m}}$, and these two algebras are equal if $S_1 = \emptyset$. The algebra $\mathbf{T}_{\mathfrak{m},S_1}$ is a local, commutative sub-algebra of $\mathrm{End}\,_{\mathcal{O}}(X_{\mathfrak{m},S_1})$. It is reduced and finite free as an $\mathcal{O}$-module. Let

$$r_{\mathfrak{m},S_1} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbf{T}_{\mathfrak{m},S_1})$$

denote the continuous lifting of $\overline{r}_{\mathfrak{m}}$ provided by proposition 3.4.4. Then $\mathbf{T}_{\mathfrak{m},S_1}$ is generated as an $\mathcal{O}$-algebra by the coefficients of the characteristic polynimials of $r_{\mathfrak{m},S_1}(\sigma)$ for $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$.

For $S_1 \subset S$, consider the deformation problem $\mathcal{S}_{S_1}$ given by

$$(F/F^+, T, \widetilde{T}, \mathcal{O}, \overline{r}_{\mathfrak{m}}, \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}, \{\mathcal{D}_v\}_{v\in T})$$

where:

- For $v \in S_a$, $\mathcal{D}_v$ will consist of all lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and

$$L_v = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), \mathrm{ad}\,\overline{r}_{\mathfrak{m}}) = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

– For $v \in S_l$, $\mathcal{D}_v$ and L_v are as described in section 2.4.1 (i.e. consists of crystalline deformations).

– For $v \in S(B)$, $\mathcal{D}_v$ consists of lifts which are $\widetilde{r}_{\widetilde{v}}$-discrete series as described in section 2.4.5. In this case L_v is also described in section 2.4.5.

– For $v \in S - S_1$, $\mathcal{D}_v$ will consist of all unramified lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and

$$L_v = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

– For $v \in S_1$, $\mathcal{D}_v$ will consist of all lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and

$$L_v = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

Also let

$$r^{\mathrm{univ}}_{\mathfrak{m},S_1} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(R^{\mathrm{univ}}_{\mathfrak{m},S_1})$$

denote the universal deformation of $\overline{r}_{\mathfrak{m}}$ of type $\mathcal{S}_{S_1}$. By proposition 3.4.4 there is a natural surjection

$$R^{\mathrm{univ}}_{S_1} \twoheadrightarrow \mathbf{T}_{\mathfrak{m},S_1}$$

such that $r^{\mathrm{univ}}_{\mathfrak{m},S_1}$ pushes forward to $r_{\mathfrak{m},S_1}$.

Theorem 5.4.1 *Keep the notation and assumptions of the start of this section. Assume also that conjecture I is true for G and G'. Then*

$$R^{\mathrm{univ}}_{\mathfrak{m},S} \xrightarrow{\sim} \mathbf{T}_{\mathfrak{m},S}$$

is an isomorphism of complete intersections.

Proof: As in section 5.3 we see that we have a commutative diagram

$$\begin{array}{ccccc} R^{\mathrm{univ}}_{\mathfrak{m},S} & \twoheadrightarrow & \mathbf{T}_{\mathfrak{m},S} & & \\ \downarrow & & \downarrow & & \\ R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} & \xrightarrow{\sim} & \mathbf{T}_{\mathfrak{m},\emptyset} & \xrightarrow{\phi} & \mathcal{O}. \end{array}$$

The lower left map is an isomorphism by theorem 3.5.1. Let ϕ_S denote the composite $\mathbf{T}_{\mathfrak{m},S} \to \mathbf{T}_{\mathfrak{m},\emptyset} \xrightarrow{\phi} \mathcal{O}$. Let $\mathfrak{c}_\emptyset(\phi)$ (resp. $\mathfrak{c}_S(\phi)$) be the ideals $\phi(\mathrm{Ann}_{\mathbf{T}_{\mathfrak{m},\emptyset}} \ker \phi)$ (resp. $\phi_S(\mathrm{Ann}_{\mathbf{T}_{\mathfrak{m},S}} \ker \phi_S)$). Also let $\wp_\emptyset$ (resp. $\wp_S$) denote the kernel of the composite $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \twoheadrightarrow \mathbf{T}_{\mathfrak{m},\emptyset} \xrightarrow{\phi} \mathcal{O}$ (resp. $R^{\mathrm{univ}}_{\mathfrak{m},S} \twoheadrightarrow \mathbf{T}_{\mathfrak{m},S} \xrightarrow{\phi_S} \mathcal{O}$).

By theorem 3.5.1 the map $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \xrightarrow{\sim} \mathbf{T}_{\mathfrak{m},\emptyset}$ is an isomorphism of complete intersections and the main theorem of [Le] implies that

$$\mathrm{lg}_{\mathcal{O}}\, \wp_\emptyset/\wp_\emptyset^2 = \mathrm{lg}_{\mathcal{O}}\, \mathcal{O}/\mathfrak{c}_\emptyset(\phi).$$

Hence by lemma 2.3.2 and proposition 5.3.5 we see that

$$\begin{array}{l} \lg_{\mathcal{O}} \wp_S/\wp_S^2 \\ \leq \lg_{\mathcal{O}} \wp_\emptyset/\wp_\emptyset^2 + \sum_{v\in S} \lg_{\mathcal{O}} H^0(\mathrm{Gal}\,(\overline{F}_{\tilde{v}}/F_{\tilde{v}}), (\mathrm{ad}\, r_{\mathfrak{m}}) \otimes_{\mathbf{T}_{\mathfrak{m},\emptyset},\phi} K/\mathcal{O}(\epsilon^{-1})) \\ \leq \lg_{\mathcal{O}} \mathcal{O}/\mathfrak{c}_S(\phi). \end{array}$$

Another application of the main theorem of [Le] tells us that $R^{\mathrm{univ}}_{\mathfrak{m},S} \to \mathbf{T}_{\mathfrak{m},S}$ is an isomorphism of complete intersections. □

5.5. *Conditional modularity lifting theorems.* — In this section we apply theorem 5.4.1 to deduce conditional modularity lifting theorems in the non-minimal case. The following theorem is proved in exactly the same way as theorem 4.4.2, except that we appeal to theorem 5.4.1 instead of theorem 3.5.1.

Theorem 5.5.1 *Let F be an imaginary CM field and let F^+ denote its maximal totally real subfield. Let $n \in \mathbf{Z}_{\geq 1}$ and let $l > n$ be a prime which is unramified in F. Let*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r.

1. *$r^c \cong r^\vee \epsilon^{1-n}$.*
2. *r is unramified at all but finitely many primes.*
3. *For all places $v|l$ of F, $r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is crystalline.*
4. *There is an element $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)}$ such that*
   - *for all $\tau \in \mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)$ we have*

$$l-1-n \geq a_{\tau,1} \geq ... \geq a_{\tau,n} \geq 0$$

or

$$l-1-n \geq a_{\tau c,1} \geq ... \geq a_{\tau c,n} \geq 0;$$

   - *for all $\tau \in \mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)$ and all $i = 1, ..., n$*

$$a_{\tau c,i} = -a_{\tau,n+1-i};$$

   - *for all $\tau \in \mathrm{Hom}\,(F,\overline{\mathbf{Q}}_l)$ above a prime $v|l$ of F,*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$

unless $i = a_{\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

5. There is a non-empty finite set S of places of F not dividing l and for each $v \in S$ a square integrable representation ρ_v of $GL_n(F_v)$ over $\overline{\mathbf{Q}}_l$ such that

$$r|^{\rm ss}_{{\rm Gal}\,(\overline{F}_v/F_v)} = r_l(\rho_v)^\vee(1-n)^{\rm ss}.$$

If $\rho_v = {\rm Sp}_{m_v}(\rho_v')$ then set

$$\widetilde{r}_v = r_l((\rho_v')^\vee|\ \ |^{(n/m_v-1)(1-m_v)/2}).$$

Note that $r|_{{\rm Gal}\,(\overline{F}_v/F_v)}$ has a unique filtration ${\rm Fil}_v^j$ such that

$${\rm gr}_v^j r|_{{\rm Gal}\,(\overline{F}_v/F_v)} \cong \widetilde{r}_v\epsilon^j$$

for $j = 0, ..., m_v - 1$ and equals (0) otherwise. We assume that $\widetilde{r}_v$ has irreducible reduction $\overline{r}_v$. Then $\overline{r}|_{{\rm Gal}\,(\overline{F}_v/F_v)}$ inherits a filtration $\overline{{\rm Fil}}_v^j$ with

$$\overline{{\rm gr}}_v^j\overline{r}|_{{\rm Gal}\,(\overline{F}_v/F_v)} \cong \overline{r}_v\epsilon^j$$

for $j = 0, ..., m_v - 1$. Finally we suppose that for $j = 1, ..., m_v$ we have

$$\overline{r}_v \not\cong \overline{r}_v\epsilon^i.$$

6. $\overline{F}^{\ker {\rm ad}\,\overline{r}}$ does not contain $F(\zeta_l)$.

7. The image $\overline{r}({\rm Gal}\,(\overline{F}/F(\zeta_l)))$ is big in the sense of definition 2.5.1.

8. The representation $\overline{r}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$ with $S \neq \emptyset$.

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and level prime to l.

Exactly as we deduced theorem 4.4.3 from theorem 4.4.2 we can deduce the following variant of theorem 5.5.1 for totally real fields.

Theorem 5.5.2 *Let F^+ be a totally real field. Let $n \in \mathbf{Z}_{\geq 1}$ and let $l > n$ be a prime which is unramified in F^+. Let*

$$r : {\rm Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r.

1. $r^\vee \cong r\epsilon^{n-1}\chi$ for some character $\chi : {\rm Gal}\,(\overline{F}^+/F^+) \longrightarrow \overline{\mathbf{Q}}_l^\times$ with $\chi(c_v)$ independent of $v|\infty$. (Here c_v denotes a complex conjugation at v.)

2. r ramifies at only finitely many primes.

3. For all places $v|l$ of F^+, $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ is crystalline.

4. There is an element $a \in (\mathbf{Z}^n)^{\mathrm{Hom}\,(F^+,\overline{\mathbf{Q}}_l)}$ such that

– for all $\tau \in \mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)$ we have

$$l - 1 - n + a_{\tau,n} \geq a_{\tau,1} \geq \ldots \geq a_{\tau,n};$$

– for all $\tau \in \mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)$ above a prime $v|l$ of F^+,

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 0$$

unless $i = a_{\tau,j} + n - j$ for some $j = 1, \ldots, n$ in which case

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 1.$$

5. There is a finite non-empty set S of places of F^+ not dividing l and for each $v \in S$ a square integrable representation ρ_v of $GL_n(F_v^+)$ over $\overline{\mathbf{Q}}_l$ such that

$$r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = r_l(\rho_v)^\vee(1-n)^{\mathrm{ss}}.$$

If $\rho_v = \mathrm{Sp}_{\,m_v}(\rho_v')$ then set

$$\widetilde{r}_v = r_l((\rho_v')^\vee|\ \ |^{(n/m_v-1)(1-m_v)/2}).$$

Note that $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ has a unique filtration $\mathrm{Fil}_v^{\,j}$ such that

$$\mathrm{gr}_v^{\,j} r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} \cong \widetilde{r}_v \epsilon^j$$

for $j = 0, \ldots, m_v - 1$ and equals (0) otherwise. We assume that $\widetilde{r}_v$ has irreducible reduction $\overline{r}_v$ such that

$$\overline{r}_v \not\cong \overline{r}_v \epsilon^j$$

for $j = 1, \ldots, m_v$. Then $\overline{r}|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ inherits a unique filtration $\overline{\mathrm{Fil}}_v^{\,j}$ with

$$\overline{\mathrm{gr}}_v^{\,j} \overline{r}|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} \cong \overline{r}_v \epsilon^j$$

for $j = 0, \ldots, m_v - 1$.

6. $(\overline{F}^+)^{\ker \mathrm{ad}\,\overline{r}}$ does not contain $F^+(\zeta_l)$.

7. The image $\overline{r}(\mathrm{Gal}\,(\overline{F}^+/F^+(\zeta_l)))$ is big in the sense of definition 2.5.1.

8. $\overline{r}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$ with $S \neq \emptyset$.

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and level prime to l.

5.6. *A conditional modularity theorem.* — We would like to apply theorems 5.5.1 and 5.5.2 in situations where one knows that $\overline{r}$ is automorphic. One such case is where $\overline{r} : \mathrm{Gal}\,(\overline{F}/F) \to GL_n(k)$ is induced from a (suitable) character over some cyclic extension. However it will be useful to have such a theorem when ρ_v is Steinberg for $v \in S$. Because the lift of $\overline{r}$ which we know to be automorphic is an automorphic induction it can not be Steinberg at any finite place (although it can be cuspidal at a finite place). Thus we have a problem in applying theorems 5.5.1 or 5.5.2 directly. We shall get round this by applying proposition 2.7.4 to construct a second lift r_1 of $\overline{r}$ which is Steinberg at $v \in S$, but which is also cuspidal at some other finite places S'. We first show that r_1 is automorphic using the places in S'. The result is that we succeed in 'raising the level' for the automorphicity of $\overline{r}$. We can then apply theorem 5.5.1 or 5.5.2 a second time. A further complication arises because we want to treat $\overline{r}$ which do not look as if they could have a lift which is cuspidal at any finite place. We will do so under an assumption that $\overline{r}$ extends to a representation of $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ which looks as if it could have a lift which is cuspidal at some finite place.

More precisely we will consider the following situation.

– $M/\mathbf{Q}$ is a Galois imaginary CM field of degree n with $\mathrm{Gal}\,(M/\mathbf{Q})$ cyclic generated by an element τ.

– $l > 1 + (n-1)((n+2)^{n/2} - (n-2)^{n/2})/2^{n-1}$ (e.g. $l > 8((n+2)/4)^{1+n/2}$) is a prime which splits completely in M and is $\equiv 1 \bmod n$.

– p is a rational prime which is inert and unramified in M.

– $q \neq l$ is a rational prime, which splits completely in M and which satisfies $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n-1$.

– $\overline{\theta} : \mathrm{Gal}\,(\overline{\mathbf{Q}}/M) \longrightarrow \overline{\mathbf{F}}_l^\times$ is a continuous character such that

 – $\overline{\theta}\overline{\theta}^c = \epsilon^{1-n}$;

 – there exists a prime $w|l$ of M such that for $i = 0, ..., n/2-1$ we have $\overline{\theta}|_{I_{\tau^i w}} = \epsilon^{-i}$;

 – if $v_1, ..., v_n$ are the primes of M above q then $\{\overline{\theta}(\mathrm{Frob}_{v_i})\} = \{\alpha_q q^{-j} : j = 0, ..., n-1\}$ for some $\alpha_q \in \overline{\mathbf{F}}_l^\times$;

 – $\overline{\theta}|_{\mathrm{Gal}\,(\overline{M}_p/M_p)} \neq \overline{\theta}^{\tau^j}|_{\mathrm{Gal}\,(\overline{M}_p/M_p)}$ for $j = 1, ..., n-1$.

Let $S(\overline{\theta})$ denote the set of rational primes above which M or $\overline{\theta}$ is ramified.

– $E/\mathbf{Q}$ is an imaginary quadratic field linearly disjoint from the Galois closure of $\overline{M}^{\ker \overline{\theta}}(\zeta_l)/\mathbf{Q}$ in which every element of $S(\overline{\theta}) \cup \{l, q, p\}$ splits; and such that the class number of E is not divisible by l.

Theorem 5.6.1 *Keep the notation and assumptions listed above. Let F/F_0 be a Galois extension of imaginary CM fields with F linearly disjoint from* *the normal closure of* $\overline{M}^{\ker \overline{\theta}}(\zeta_l)$ *over* $\mathbf{Q}$. *Assume that* l *is unramified in* F *and that there is a prime* $v_{p,0}$ *of* F_0 *split above* p. *Let*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ *denote the semisimplification of the reduction of* r.

1. $\overline{r} \cong \mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{F}/F)}_{\mathrm{Gal}\,(\overline{F}/FM)} \overline{\theta}|_{\mathrm{Gal}\,(\overline{F}/FM)}$.
2. $r^c \cong r^\vee \epsilon^{1-n}$.
3. r *ramifies at only finitely many primes.*
4. *For all places* $v|l$ *of* F, $r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ *is crystalline.*
5. *For all* $\tau \in \mathrm{Hom}\,(F, \overline{\mathbf{Q}}_l)$ *above a prime* $v|l$ *of* F,

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau, F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1$$

for $i = 0, ..., n-1$ *and* $= 0$ *otherwise.*

6. *There is a place* v_q *of* F *above* q *such that* $(\#k(v_q))^j \not\equiv 1 \bmod l$ *for* $j = 1, ..., n$, *and such that* $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}$ *is unramified, and such that* $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}(\mathrm{Frob}_{v_q})$ *has eigenvalues* $\{\alpha(\#k(v_q))^j : j = 0, ..., n-1\}$ *for some* $\alpha \in \overline{\mathbf{Q}}_l^\times$.

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r *is automorphic over* F *of weight* 0 *and type* $\{\mathrm{Sp}_n(1)\}_{\{v_q\}}$ *and level prime to* l.

Proof: Replacing F by EF if necessary we may suppose that $F \supset E$ (see lemma 4.2.2).

Choose a continuous character

$$\theta : \mathrm{Gal}\,(\overline{M}/M) \longrightarrow \mathcal{O}_{\overline{\mathbf{Q}}_l}^\times$$

such that

- θ lifts $\overline{\theta}$;
- $\theta^{-1} = \epsilon^{n-1}\theta^c$;
- for $i = 0, ..., n/2-1$ we have $\theta|_{I_{M_{\sigma^i w}}} = \epsilon^{-i}$; and
- $l \not| \#\theta(I_v)$ for all places $v|p$ of M.

(See lemma 4.1.6.) We can extend $\theta|_{\mathrm{Gal}\,(\overline{E}/EM)}$ to a continuous homomorphism

$$\theta : \mathrm{Gal}\,(\overline{E}/(EM)^+) \longrightarrow \mathcal{G}_1(\mathcal{O}_{\overline{\mathbf{Q}}_l})$$

with $\nu \circ \theta = \epsilon^{1-n}$. We will let $\overline{\theta}$ also denote the reduction

$$\overline{\theta} : \mathrm{Gal}\,(\overline{E}/(EM)^+) \longrightarrow \mathcal{G}_1(\overline{\mathbf{F}}_l)$$

of θ. Consider the pairs $\mathrm{Gal}\,(\overline{E}/(EM)^+) \supset \mathrm{Gal}\,(\overline{E}/(EM))$ and $\mathrm{Gal}\,(\overline{E}/\mathbf{Q}) \supset \mathrm{Gal}\,(\overline{E}/E)$. Set

$$r_0 = \mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{E}/\mathbf{Q}),\epsilon^{1-n}}_{\mathrm{Gal}\,(\overline{E}/(EM)^+)}\theta : \mathrm{Gal}\,(\overline{E}/\mathbf{Q}) \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbf{Q}}_l}).$$

Note also that

$$r_0|_{\mathrm{Gal}\,(\overline{E}/E)} = ((\mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{E}/\mathbf{Q})}_{\mathrm{Gal}\,(\overline{E}/M)}\theta)|_{\mathrm{Gal}\,(\overline{E}/E)}, \epsilon^{1-n}).$$

By proposition 2.7.4 there is a continuous homomorphism

$$r_1 : \mathrm{Gal}\,(\overline{E}/\mathbf{Q}) \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbf{Q}}_l})$$

with the following properties.

- r_1 lifts $\mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{E}/\mathbf{Q}),\epsilon^{1-n}}_{\mathrm{Gal}\,(\overline{E}/(EM)^+)}\overline{\theta}$.
- $\nu \circ r_1 = \epsilon^{1-n}$.
- For all places $w|l$ of E, $r_1|_{\mathrm{Gal}\,(\overline{E}_w/E_w)}$ is crystalline.
- For all $\tau \in \mathrm{Hom}\,(E, \overline{\mathbf{Q}}_l)$ corresponding to prime $w|l$,

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}\,^i(r_1 \otimes_{\tau,E_w} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{E}_w/E_w)} = 1$$

for $i = 0, ..., n-1$ and $= 0$ otherwise.
- $r_1|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_{v_q}/E_{v_q})}$ is unramified and $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_{v_q}/E_{v_q})}(\mathrm{Frob}_{v_q|_E})$ has eigenvalues $\{\alpha q^{-j} :\ j = 0, ..., n-1\}$ for some $\alpha \in \overline{\mathbf{Q}}_l^{\times}$.
- $r_1|_{\mathrm{Gal}\,(\overline{E}_{v_p}/E_{v_p})}$ is an unramified twist of $\mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}_{\mathrm{Gal}\,(\overline{\mathbf{Q}}_p/M_p)}\theta|_{\mathrm{Gal}\,(\overline{\mathbf{Q}}_p/M_p)}$.

Let v_p be a prime of F above $v_{p,0}$ and let $F_1 \subset F$ denote the fixed field of the decomposition group of v_p in $\mathrm{Gal}\,(F/F_0)$. Thus $v_p|_{F_1}$ is split over p and F/F_1 is soluble.

The restriction $r_0|_{\mathrm{Gal}\,(\overline{E}/F_1)}$ is automorphic of weight 0, level prime to l and type $\{\rho_p\}_{\{v_p|_{F_1}\}}$, for a suitable cuspidal representation ρ_p (by theorem 4.2 of [AC]). Applying lemma 2.7.5 and theorem 5.5.1 we deduce that $r_1|_{\mathrm{Gal}\,(\overline{F}/F_1)}$ is automorphic of weight 0 and type $\{\rho_p\}_{\{v_p|_{F_1}\}}$ and level prime to l. It follows from corollary VII.1.11 of [HT] that $r_1|_{\mathrm{Gal}\,(\overline{F}/F_1)}$ is also automorphic of weight 0 and type $\{\mathrm{Sp}\,_n(1)\}_{\{v_q|_{F_1}\}}$ and level prime to l. (The only tempered representations π of $GL_n(F_{1,v_q|_{F_1}})$ for which $r_l(\pi)^\vee(1-n)^{\mathrm{ss}}$ unramified and $r_l(\pi)^\vee(1-n)^{\mathrm{ss}}(\mathrm{Frob}_{v_q|_{F_1}})$ has eigenvalues of the form $\{\alpha q^{-j} :\ j = 0, ..., n-1\}$ are unramified twists of $\mathrm{Sp}\,_n(1)$.) From theorem 4.2 of [AC] we deduce that

$r_1|_{\mathrm{Gal}\,(\overline{F}/F)}$ is automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v_q\}}$ and level prime to l. (The base change must be cuspidal as it is square integrable at one place.)

Finally we again apply theorem 5.5.1 to deduce that r is automorphic of weight 0 and type $\{\rho_p\}_{\{v_p\}}$ and level prime to l. The verification that $\overline{r}(G_{F^+(\zeta_l)})$ is big is exactly as above. $\square$

We also have a version for totally real fields.

Theorem 5.6.2 *Keep the notation and assumptions listed at the start of this section. Let F^+/F_0^+ be a Galois extension of totally real fields with F^+ linearly disjoint from the Galois closure of $E(\zeta_l)\overline{M}^{\ker \overline{\theta}}$ over $\mathbf{Q}$. Suppose that that l is unramified in F^+ and that there is a prime $v_{p,0}$ of F_0^+ split over p. Let*

$$r : \mathrm{Gal}\,(\overline{F^+}/F^+) \longrightarrow GL_n(\overline{\mathbf{Q}}_l)$$

be a continuous representation such that

- $\overline{r} \cong (\mathrm{Ind}\,_{\mathrm{Gal}\,(\overline{\mathbf{Q}}/M)}^{\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})}\overline{\theta})|_{\mathrm{Gal}\,(\overline{\mathbf{Q}}/F^+)}$;
- $r^\vee \cong r\epsilon^{n-1}$;
- *r is unramified at all but finitely many primes;*
- *For all places $v|l$ of F^+, $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ is crystalline.*
- *For all $\tau \in \mathrm{Hom}\,(F^+, \overline{\mathbf{Q}}_l)$ above a prime $v|l$ of F^+,*

$$\dim_{\overline{\mathbf{Q}}_l} \mathrm{gr}\,^i(r \otimes_{\tau, F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 1$$

for $i = 0, ..., n-1$ and $= 0$ otherwise.

- *There is a place $v_q|q$ of F^+ such that*
  - $\#k(v_q)^j \not\equiv 1 \bmod l$ *for $j = 1, ..., n-1$,*
  - $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}^+/F_{v_q}^+)}$ *is unramified, and*
  - $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}^+/F_{v_q}^+)}(\mathrm{Frob}_{v_q})$ *has eigenvalues* $\{\alpha(\#k(v_q))^j : j = 0, ..., n-1\}$

for some $\alpha \in \overline{\mathbf{Q}}_l^\times$.

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic over F^+ of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v_q\}}$ and level prime to l.

Proof: Apply theorem 5.6.1 to $F = F^+E$ and use lemma 4.3.3. $\square$

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APPENDIX A: The level raising operator after Russ Mann.

In this appendix we will explain Russ Mann's proof of lemma 5.1.6 and proposition 5.1.7. A preliminary write-up of most of the arguments can be found in [Man2], but as Russ has left academia it seems increasingly unlikely that he will finish [Man2]. Hence this appendix. Russ actually found more general results concerning level raising for forms of level greater than 1, which we do not report on here. We stress that the arguments of this appendix are entirely due to Russ Mann, though we of course take responsibility for any errors in their presentation.

Write B_n for the Borel subgroup of GL_n consisting of upper triangular matrices and write N_n for its unipotent radical. Also write T_n for the maximal torus in GL_n consisting of diagonal matrices and write P_n for the subgroup of GL_n consisting of matrices with last row $(0, ..., 0, 1)$.

Let F_w be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_{F_w}$. Let $w : F_w^\times \twoheadrightarrow \mathbf{Z}$ denote the valuation, let ϖ_w denote a uniformiser of $\mathcal{O}_{F_w}$ and let $q_w = \#\mathcal{O}_{F_w}/(\varpi_w)$. Also let $\mathcal{O}$ denote the subring of $\mathbf{C}$ generated by $q_w^{-1/2}$ and all p-power roots of 1. Let S_n denote the symmetric group on n letters and set

$$R_n^+ = \mathcal{O}[X_1, ..., X_n]^{S_n} \subset R_n = \mathcal{O}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^{S_n},$$

where S_n permutes the variables X_i. Sometimes we will want to consider R_n and R_{n-1} at the same time. To make the notation clearer we will write $R_{n-1} = \mathcal{O}[Y_1^{\pm 1}, ..., Y_{n-1}^{\pm 1}]^{S_{n-1}}$ and $R_{n-1}^+ = \mathcal{O}[Y_1, ..., Y_{n-1}]^{S_{n-1}}$. We will also set

$$R_{n-1}^\wedge = \mathcal{O}[[Y_1, ..., Y_{n-1}]]^{S_{n-1}}$$

and $R_{n-1}^{\leq m}$ to equal to the $\mathcal{O}$-submodule of R_{n-1}^+ consisting of polynomials of degree $\leq m$ in each variable separately.

Let $\alpha_j = \varpi_w 1_j \oplus 1_{n-j}$ and let $T^{(j)}$ denote the double coset

$$T^{(j)} = GL_n(\mathcal{O}_{F_w})\alpha_j GL_n(\mathcal{O}_{F_w}).$$

Let $GL_n(\mathcal{O}_{F_w})^+$ denote the sub-semigroup of $GL_n(F_w)$ consisting of matrices with entries in $\mathcal{O}_{F_w}$. Then

$$\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)^+/GL_n(\mathcal{O}_{F_w})] = \mathcal{O}[T^{(1)}, T^{(2)}, ..., T^{(n)}]$$

and

$$\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] = \mathcal{O}[T^{(1)}, T^{(2)}, ..., T^{(n)}, (T^{(n)})^{-1}].$$

Define $^\sim$ from $\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ to itself by

$$[GL_n(\mathcal{O}_{F_w})gGL_n(\mathcal{O}_{F_w})]^\sim = [GL_n(\mathcal{O}_{F_w})g^{-1}GL_n(\mathcal{O}_{F_w})].$$

Then $(T^{(j)})^\sim = (T^{(n)})^{-1}T^{(n-j)}$.

There is an isomorphism (a certain normalisation of the the Satake isomorphism)

$$S : \mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \xrightarrow{\sim} R_n$$

which sends $T^{(j)}$ to $q_w^{j(1-j)/2}$ times the j^{th} elementary symmetric function in the X_i's (i.e. to the sum of all products of j distinct X_i's). We have

$$S(\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)^+/GL_n(\mathcal{O}_{F_w})]) = R_n^+$$

and

$$S(T^\sim)(X_1, ..., X_n) = S(T)(q_w^{n-1}X_1^{-1}, ..., q_w^{n-1}X_n^{-1}).$$

If we write

$$\mathcal{O}[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$$

for the submodule of $\mathcal{O}[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]$ spanned by the double cosets

$$GL_{n-1}(\mathcal{O}_{F_w})\mathrm{diag}(t_1, ..., t_{n-1})GL_{n-1}(\mathcal{O}_{F_w}),$$

where $m \geq w(t_1) \geq ... \geq w(t_{n-1}) \geq 0$, then

$$S(\mathcal{O}[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}) = (\mathcal{O}[Y_1, ..., Y_{n-1}]^{S_{n-1}})^{\leq m}.$$

Let $U_1(w^m)$ denote the subgroup of $GL_n(\mathcal{O}_{F_w})$ consisting of elements which reduce modulo ϖ_w^m to an element of $P_n(\mathcal{O}_{F_w}/(\varpi_w^m))$. For $j = 1, ..., n-1$ let

$$U^{(j)} = P_n(\mathcal{O}_{F_w})\alpha_j P_n(\mathcal{O}_{F_w}).$$

Note that $U^{(j)}/P_n(\mathcal{O}_{F_w})$ has finite cardinality. If π is a smooth representation of $GL_n(F_w)$ and if $m \in \mathbf{Z}_{\geq 1}$ then

- the operators $U^{(j)}$ on $\pi^{P_n(\mathcal{O}_{F_w})}$ commute, and
- the action of $U^{(j)}$ preserves $\pi^{U_1(w^m)}$ and in fact acts the same way as

$$U_1(w^m)\alpha_j U_1(w^m)$$

  on this space.

(This is proved by writing down explicit coset decompositions, see for instance proposition 4.1 of [Man1] .)

Let A be an $\mathcal{O}$-module and suppose that

$$T = \sum_i a_i GL_{n-1}(\mathcal{O}_{F_w})g_i GL_{n-1}(\mathcal{O}_{F_w})$$

is in $A[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]$. Define

$$V(T)=\sum_i a_i|\det g_i|^{n-1/2}P_n(\mathcal{O}_{F_w})\begin{pmatrix} g_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w}).$$

Note that if $h\in GL_{n-1}(F_w)^+$ and

$$GL_{n-1}(\mathcal{O}_{F_w})h^{-1}GL_{n-1}(\mathcal{O}_{F_w})=\coprod_j h_jGL_{n-1}(\mathcal{O}_{F_w})$$

then

$$P_n(\mathcal{O}_{F_w})\begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w})=\coprod_j\begin{pmatrix} h_j & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w}).$$

Similarly if $m\in\mathbf{Z}_{\geq 1}$ and if

$$T=\sum_i a_iGL_{n-1}(\mathcal{O}_{F_w})g_iGL_{n-1}(\mathcal{O}_{F_w})$$

is in $A[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$ define

$$V_m(T)=\sum_i a_i|\det g_i|^{n-1/2}U_1(w^m)\begin{pmatrix} g_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w}).$$

Note that if $h\in GL_{n-1}(F_w)^+$ is such that $GL_{n-1}(\mathcal{O}_{F_w})hGL_{n-1}(\mathcal{O}_{F_w})$ lies in $A[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$, and if

$$GL_{n-1}(\mathcal{O}_{F_w})h^{-1}GL_{n-1}(\mathcal{O}_{F_w})=\coprod_j h_jGL_{n-1}(\mathcal{O}_{F_w})$$

then

$$U_1(w^m)\begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w})=\coprod_j\begin{pmatrix} h_j & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w}).$$

We deduce that if π is any smooth representation of $GL_n(F_w)$ and if $T\in A[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$ then $V(T)$ preserves the space $\pi^{U_1(w^m)}$ and acts on it via $V_m(T)$. In the case $A=R_n$ the map V_m induces a map, which we will also denote V_m, from the module

$$R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$$

to $\mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ given by the formula

$$\begin{aligned}&V_m(\textstyle\sum_i a_i[GL_{n-1}(\mathcal{O}_{F_w})g_iGL_{n-1}(\mathcal{O}_{F_w})])\\ &=\textstyle\sum_i|\det g_i|^{n-1/2}\left[U_1(w^m)\begin{pmatrix} g_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}GL_n(\mathcal{O}_{F_w})\right]\circ S^{-1}(a_i).\end{aligned}$$

Proposition 5.2 of [Man1] says that the set of

$$V_m(GL_{n-1}(\mathcal{O}_{F_w})\mathrm{diag}(t_1, ..., t_{n-1})GL_{n-1}(\mathcal{O}_{F_w})),$$

where $t \in T_{n-1}(F_w)/T_{n-1}(\mathcal{O}_{F_w})$ with $m \geq w(t_1) \geq ... \geq w(t_{n-1}) \geq 0$ is a basis of $\mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ as a right R_n-module. Hence the map V_m from

$$R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$$

to $\mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ is an isomorphism of free R_n-modules.

Let

$$\psi : F_w \longrightarrow \mathcal{O}^\times$$

be a continuous character with kernel $\mathcal{O}_{F_w}$. We will also think of ψ as a character of $N_n(F_w)$ by setting

$$\psi(u) = \psi(u_{1,2} + u_{2,3} + ... + u_{n-1,n}).$$

If A is an $\mathcal{O}$-algebra we will write $\mathcal{W}_n(A, \psi)$ for the set of functions

$$W : GL_n(F_w) \longrightarrow A$$

such that

- $W(ug) = \psi(u)W(g)$ for all $g \in GL_n(F_w)$ and $u \in N_n(F_w)$,
- and W is invariant under right translation by some open subgroup of $GL_n(F_w)$.

Thus $\mathcal{W}_n(A, \psi)$ is a smooth representation of $GL_n(F_w)$ (acting by right translation).

There is a unique element $W_n^0(\psi) \in \mathcal{W}_n(R_n, \psi)^{GL_n(\mathcal{O}_{F_w})}$ such that

- $W_n^0(\psi)(1_n) = 1$ and
- $TW_n^0(\psi) = S(T)W_n^0(\psi)$ for all $T \in \mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$.

Moreover if the last row of g is integral then $W_n^0(\psi)(g) \in R_n^+$. (These facts are proved exactly as in [Sh].)

Suppose again that A is an $\mathcal{O}$-algebra. If $W \in \mathcal{W}_n(A, \psi)^{P_n(\mathcal{O}_{F_w})}$ we heuristically define $\Phi(W) \in A \otimes_{\mathcal{O}} R_{n-1}^\wedge = A[[Y_1, ..., Y_{n-1}]]^{S_{n-1}}$ by

$$\Phi(W) = \int_{N_{n-1}(F_w)\backslash GL_{n-1}(F_w)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_{n-1}^0(\psi^{-1})(g)|\det g|^{s-n+1/2}dg\bigg|_{s=0}$$

where the implies Haar measures give $GL_{n-1}(\mathcal{O}_{F_w})$ and $N_{n-1}(\mathcal{O}_{F_w})$ volume 1. Rigorously one can for instance set

$$\Phi(W) = \sum_t W\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} W_{n-1}^0(\psi^{-1})(t)|\det t|^{s-n+1/2}|t_1|^{2-n}|t_2|^{4-n}...|t_{n-1}|^{n-2}$$

where $t = \mathrm{diag}(t_1, ..., t_{n-1})$ runs over elements of $T_{n-1}(F_w)/T_{n-1}(\mathcal{O}_{F_w})$ with

$$w(t_1) \geq w(t_2) \geq ... \geq w(t_{n-1}) \geq 0.$$

For such t the value $W^0_{n-1}(\psi^{-1})(t)$ is a homogeneous polynomial in the Y_i's of degree $w(\det t)$ and these polynomials are linearly independent over A for $t \in T_{n-1}(F_w)/T_{n-1}(\mathcal{O}_{F_w})$ with $w(t_1) \geq w(t_2) \geq ... \geq w(t_{n-1}) \geq 0$. (As in [Sh].) In particular if $W \in \mathcal{W}_n(A, \psi)^{P_n(\mathcal{O}_{F_w})}$ then $\Phi(W)$ determines $W|_{P_n(F_w)}$. As in section (1.4) of [JS2] we see that

$$\Phi(W^0_n(\psi)) = \prod_{i,j}(1 - X_iY_j)^{-1}.$$

Fix an embedding $\imath : R_n \hookrightarrow \mathbf{C}$. There is a unique irreducible smooth representation π of $GL_n(F_w)$ such that $\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ acts on $\pi^{GL_n(\mathcal{O}_{F_w})}$ via $\imath \circ S$. Moreover there is an embedding $\pi \hookrightarrow \mathcal{W}_n(\mathbf{C}, \psi)$ which is unique up to $\mathbf{C}^\times$-multiples. It follows from [Sh] that $\imath W^0_n(\psi)$ is in the image of π. It follows from sections (3.5) and (4.2) of [JPSS] that

$$\Phi : (R_n[GL_n(F_w)]W^0_n(\psi))^{P_n(\mathcal{O}_{F_w})} \hookrightarrow \prod_{i,j}(1 - X_iY_j)^{-1}R_n[Y_1, ..., Y_{n-1}]^{S_{n-1}}.$$

From corollary 3.5 of [Man1] we see also see that

$$\dim_{\mathbf{C}}(R_n[GL_n(F_w)]W^0_n(\psi))^{U_1(w^m)}) \otimes_{R_n,\imath} \mathbf{C} \leq \dim_{\mathbf{C}} \pi^{U_1(w^m)} = \begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix}.$$

If $W \in (R_n[GL_n(F_w)]W^0_n(\psi))^{P_n(\mathcal{O}_{F_w})}$ and $\Phi(W) = 1$ then we see that $W|_{P_n(F_w)}$ is supported on $N_n(F_w)P_n(\mathcal{O}_{F_w})$ and that $W(1_n) = 1$. Thus we have $(U^{(j)}W)|_{P_n(F_w)} = 0$. (Recall that we only have to check this at elements $\mathrm{diag}(t_1, ..., t_{n-1}, 1)$ and that any element of $\mathcal{W}_n(R_n, \psi)$ will vanish at $\mathrm{diag}(t_1, ..., t_{n-1}, 1)$ unless $w(t_i) \geq 0$ for all i. To check at the remaining diagonal matrices one uses the explicit single coset decomposition in proposition 4.1 of [Man1].) Hence $\Phi(U^{(j)}W) = 0$ and so $U^{(j)}W = 0$.

Recall that if $h \in GL_{n-1}(F_w)^+$ and

$$GL_{n-1}(\mathcal{O}_{F_w})h^{-1}GL_{n-1}(\mathcal{O}_{F_w}) = \coprod_j h_jGL_{n-1}(\mathcal{O}_{F_w})$$

then

$$P_n(\mathcal{O}_{F_w}) \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}) = \coprod_j \begin{pmatrix} h_j & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}).$$

From this and a simple change of variable in the integral defining Φ, we see that if T is in $A[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]$ and f is in $\mathcal{W}_n(A,\psi)^{GL_n(\mathcal{O}_{F_w})}$ then

$$\Phi(V(T)f) = S(T)\Phi(f).$$

Thus we have

$$\begin{array}{ccc} R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m} & T & \\ \downarrow & \downarrow & \\ (R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)} & V_m(T)W_n^0(\psi) & W \\ \downarrow & & \downarrow \\ \prod_{i,j}(1-X_iY_j)^{-1}R_n[Y_1,...,Y_{n-1}]^{S_{n-1}} & & \Phi(W). \end{array}$$

The composite sends

$$T \longmapsto S(T)\prod_{i,j}(1-X_iY_j)^{-1}.$$

The composite is an isomorphism to its image:

$$\prod_{i,j}(1-X_iY_j)^{-1}(R_n[Y_1,...,Y_{n-1}]^{S_{n-1}})^{\leq m},$$

which is a direct summand of $\prod_{i,j}(1-X_iY_j)^{-1}R_n[Y_1,...,Y_{n-1}]^{S_{n-1}}$ and which is free over R_n of rank

$$\binom{m+n-1}{n-1}.$$

As

$$\dim_{\mathbf{C}}(R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)})\otimes_{R_n,\imath}\mathbf{C} \leq \binom{m+n-1}{n-1},$$

we deduce that

$$\begin{array}{ll} & R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m} \\ \xrightarrow{\sim} & \mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \\ \xrightarrow{\sim} & (R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)} \\ \xrightarrow{\sim} & \prod_{i,j}(1-X_iY_j)^{-1}(R_n[Y_1,...,Y_{n-1}]^{S_{n-1}})^{\leq m}. \end{array}$$

Lemma 5.1.6 follows immediately from this.

Let θ denote the element of

$$\mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

which is $V_n(\prod_{i,j}(1-X_iY_j))$. Then

$$\Phi(\theta W_n^0(\psi)) = 1.$$

Moreover $U^{(j)}\theta W_n^0(\psi)=0$ and so $U^{(j)}\theta=0$ for $j=1,...,n-1$. Thus θ satisfies the first three parts of proposition 5.1.7.

We now turn to the proof of the final part of proposition 5.1.7. Write

$$\theta=\sum_{\underline{a}}[U_1(w^n)\mathrm{diag}(\varpi_w^{-a_1},...,\varpi_w^{-a_{n-1}},1)GL_n(\mathcal{O}_{F_w})]T_{\underline{a}}$$

where $T_{\underline{a}}\in\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ and where $\underline{a}=(a_1,...,a_{n-1})$ runs over elements of $\mathbf{Z}^{n-1}$ with

$$n\geq a_1\geq...\geq a_{n-1}\geq 0.$$

As

$$\sum_{\underline{a}}S(T_{\underline{a}})S(GL_{n-1}(\mathcal{O}_{F_w})\mathrm{diag}(\varpi_w^{a_1},...,\varpi_w^{a_{n-1}})GL_{n-1}(\mathcal{O}_{F_w}))=\prod_{i,j}(1-X_iY_j)$$

we see that

$$S(T_{(n,...,n)})=(X_1...X_n)^{n-1},$$

i.e. $T_{(n,...,n)}=q_w^{n(n-1)^2/2}(T^{(n)})^{n-1}$. Let $\eta=1_{n-1}\oplus\varpi_w^n$ and define $\widehat{\theta}$ as we did just before proposition 5.1.7. Thus we have

$$\widehat{\theta}=\sum_{\underline{a}}(T^{(n)})^{-n}T_{\underline{a}}[GL_n(\mathcal{O}_{F_w})\mathrm{diag}(\varpi_w^{n-a_1},...,\varpi_w^{n-a_{n-1}},1)U_1(w^n)].$$

Again π denote the $GL_n(F_w)$-subrepresentation of $\mathcal{W}_n(\mathbf{C},\psi)$ generated by $\imath W_n^0(\psi)$. Define $\widetilde{\imath}:R_n\hookrightarrow\mathbf{C}$ to be the $\mathcal{O}$-linear map sending X_i to $q_w^{n-1}\imath(X_i)^{-1}$. Let $\widetilde{\pi}$ denote the $GL_n(F_w)$-subrepresentation of $\mathcal{W}_n(\mathbf{C},\psi^{-1})$ generated by $\widetilde{\imath}(W_n^0(\psi^{-1}))$. Then $\widetilde{\pi}$ is the contragredient of π. Write gen_n for the compact induction $\text{c-Ind}_{N_n(F_w)}^{P_n(F_w)}\mathbf{C}(\psi)$. It follows from proposition 3.2 and lemma 4.5 of [BZ] that gen embeds in $\pi|_{P_n(F_w)}$ and in $\widetilde{\pi}|_{P_n(F_w)}$. Moreover it follows from proposition 3.8 and lemma 4.5 of [BZ] that any $P_n(F_w)$ bilinear form

$$\langle\ ,\ \rangle:\pi\times\widetilde{\pi}\longrightarrow\mathbf{C}$$

restricts non-trivially to $\mathrm{gen}_n\times\mathrm{gen}_n$. Hence there is a unique such bilinear form up to scalar multiples and so any $P_n(F_w)$-bilinear pairing $\pi\times\widetilde{\pi}\rightarrow\mathbf{C}$ is also $GL_n(F_w)$-bilinear. Such a pairing is given by

$$\langle W,\widetilde{W}\rangle=\int_{N_n(F_w)\backslash P_n(F_w)}W(g)\widetilde{W}(g)|\det g|^s dg\Big|_{s=0}.$$

Here we use a Haar measure on $N_n(F_w)$ giving $N_n(\mathcal{O}_{F_w})$ volume 1 and a right Haar measure on $P_n(F_w)$ giving $P_n(\mathcal{O}_{F_w})$ volume 1. The integral may not converge for $s=0$, but in its domain of convergence it is a rational function of q_w^s and so has meromorphic continuation to the whole complex plane.

We will complete the proof of proposition 5.1.7 by evaluating

$$\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath}W_n^0(\psi^{-1})\rangle$$

in two ways. Firstly moving the $\widehat{\theta}$ to the other side of the pairing we obtain

$$\begin{array}{l}[GL_n(\mathcal{O}_{F_w}) : U_1(w^n)] \sum_{\underline{a}} \widetilde{\imath} \circ S(\widetilde{T}_{\underline{a}}(T^{(n)})^n) \\ \langle \imath\theta W_n^0(\psi), \widetilde{\imath}[U_1(w^n)\mathrm{diag}(\varpi_w^{a_1-n},\ldots,\varpi_w^{a_{n-1}-n},1)GL_n(\mathcal{O}_{F_w})]W_n^0(\psi^{-1})\rangle.\end{array}$$

The restriction $(\theta W_n^0(\psi))|_{P_n(F_w)}$ is supported on $N_n(F_w)P_n(\mathcal{O}_{F_w})$ and equals 1 on $P_n(\mathcal{O}_{F_w})$. Thus $\langle \imath\theta W_n^0(\psi), \widetilde{W}\rangle$ simply equals $\widetilde{W}(1_n)$. We deduce that

$$\begin{array}{c}\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath}W_n^0(\psi^{-1})\rangle = (q_w^n-1)q_w^{n(n-1)} \sum_{\underline{a}} \widetilde{\imath} \circ S(\widetilde{T}_{\underline{a}}(T^{(n)})^n) \\ \widetilde{\imath}([U_1(w^n)\mathrm{diag}(\varpi_w^{a_1-n},\ldots,\varpi_w^{a_{n-1}-n},1)GL_n(\mathcal{O}_{F_w})]W_n^0(\psi^{-1}))(1_n).\end{array}$$

The terms of this sum are zero except for the term $a_1=\ldots=a_{n-1}=n$ which gives

$$(q_w^n-1)q_w^{n(n-1)}\widetilde{\imath}S(q_w^{n(n-1)^2/2}T^{(n)}),$$

i.e.

$$(q_w^n-1)q_w^{(n+2)n(n-1)/2}\imath(X_1\ldots X_n)^{-1}.$$

On the other hand

$$\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath}W_n^0(\psi^{-1})\rangle$$

equals

$$\imath(S(\widehat{\theta})\theta)\langle \imath W_n^0(\psi), \widetilde{\imath}W_n^0(\psi^{-1})\rangle.$$

We consider the integral

$$\int_{N_n(F_w)\backslash P_n(F_w)} W(g)\widetilde{W}(g)|\det g|^s dg$$

with the Haar measures described above. It equals

$$\sum_t \imath(W_n^0(\psi)(t))\widetilde{\imath}(W_n^0(\psi^{-1})(t))|t_1|^{2-n+s}|t_2|^{4-n+s}\ldots|t_n|^{n+s},$$

where the sum runs over $t=\mathrm{diag}(t_1,\ldots,t_n)\in T_n(F_w)/T_n(\mathcal{O}_{F_w})$ with

$$w(t_1)\geq w(t_2)\geq\ldots\geq w(t_n)=0.$$

Because $\imath(W_n^0(\psi)(t))\widetilde{\imath}(W_n^0(\psi^{-1})(t))$ is invariant under the multiplication of t by an element of $F_w^\times$ this in turn equals

$$(1-q_w^{-n(s+1)})\sum_t \imath(W_n^0(\psi)(t))\widetilde{\imath}(W_n^0(\psi^{-1})(t))|t_1|^{2-n+s}|t_2|^{4-n+s}...|t_n|^{n+s},$$

where now the sum runs over $t=\mathrm{diag}(t_1,...,t_n)\in T_n(F_w)/T_n(\mathcal{O}_{F_w})$ with

$$w(t_1)\geq w(t_2)\geq ...\geq w(t_n)\geq 0.$$

This in turn equals $(1-q_w^{-n(s+1)})$ times

$$\int_{N_n(F_w)\backslash GL_n(F_w)} \imath(W_n^0(\psi)(g))\widetilde{\imath}(W_n^0(\psi^{-1})(g))\varphi((0,...,0,1)g)|\det g|^{1+s}dg,$$

where φ is the characteristic function of $\mathcal{O}_{F_w}^n$ and where we use the Haar measures on $N_n(F_w)$ (resp. $GL_n(F_w)$) which give $N_n(\mathcal{O}_{F_w})$ (resp. $GL_n(\mathcal{O}_{F_w})$) volume 1. As in proposition 2 of [JS1] this becomes

$$(1-q_w^{-n(s+1)})\prod_{i=1}^n\prod_{j=1}^n(1-\imath(X_i/X_j)q_w^{-(1+s)})^{-1}.$$

Thus

$$\langle \imath\widehat{\theta}\theta W_n^0(\psi),\widetilde{\imath}W_n^0(\psi^{-1})\rangle=\imath(S(\widehat{\theta})\theta)(1-q_w^{-n})\prod_{i=1}^n\prod_{j=1}^n(1-\imath(X_i/X_j)q_w^{-1})^{-1}.$$

Thus we conclude that

$$S(\widehat{\theta}\theta)=q_w^{n^2(n-1)/2}(X_1...X_n)^{-(n+1)}\prod_{i=1}^n\prod_{j=1}^n(q_wX_i-X_j),$$

and we have completed the proof of proposition 5.1.7.

APPENDIX B: Unipotent representations of $GL(n, F)$ in the quasi-banal case.

By M.-F.Vigneras

Let F be a local non archimedean field of residual characteristic p and let R be an algebraically closed field of characteristic 0 or $\ell > 0$ different from p. Let $G = GL(n, F)$. The category $\mathrm{Mod}_R G$ of (smooth) R-representations of G is equivalent to the category of *right* modules $\mathcal{H}_R(G)$ for the global Hecke algebra (the convolution algebra of locally constant functions $f : G \to R$ with compact support, isomorphic to the opposite algebra by $f(g) \to f(g^{-1})$.)

$$\mathrm{Mod}_R G \simeq \mathrm{Mod}\mathcal{H}_R(G).$$

Definitions. We are in the *quasi-banal case* when the order of the maximal compact subgroup of G is invertible in R (the *banal case*), or when $q = 1$ in R and the characteristic of R is $\ell > n$ (the *limit case*).

A *block* of $\mathrm{Mod}_R G$ is an abelian subcategory of $\mathrm{Mod}_R G$ which is a direct factor of $\mathrm{Mod}_R G$ and is minimal for this property. One proves that $\mathrm{Mod}_R G$ is a product of blocks [V2, III.6]. The *unipotent block* $\mathcal{B}_{R,1}(G)$ is the block containing the trivial representation. An R-representation of G is *unipotent* if it belongs to the unipotent block.

Notations. Let $I, B = TU$ be a standard Iwahori, Borel, diagonal, stritly upper triangular subgroup of G, T_o the maximal compact subgroup of T, I_p the pro-p-radical of I. The functor $\mathrm{Ind}_B^G : \mathrm{Mod}_R B \to \mathrm{Mod}_R G$ is the normalised induction. The group I has a normal subgroup I^ℓ of pro-order prime to ℓ and a finite ℓ subgroup I_ℓ such that $I = I^\ell I_\ell$. To get a uniform notation, we set $I^\ell = I, I_\ell = \{1\}$ when the characteristic of R is 0. We have $I = I^\ell, I_\ell = \{1\}$ in the banal case and $I \neq I^\ell, I_\ell \neq \{1\}$ in the limit case. Let $\mathrm{Mod}H_R(G, I)$ be the category of *right* modules for the Iwahori Hecke algebra (isomorphic to its opposite)

$$H_R(G, I) := \mathrm{End}_{RG} R[I\backslash G] \simeq_R R[I\backslash G/I].$$

Let $\mathrm{Mod}_R(G, I)$ be the category of R-representations of G generated by their I-invariant vectors.

1 Theorem *In the quasi-banal case,*

1) The category $\mathrm{Mod}_R(G,I)$ *is stable by subquotients.*

2) For any $V \in \mathrm{Mod}_R(G,I)$*, one has* $V^{I_p} = V^I$*, in particular* $R[I\backslash G]$ *is projective in* $\mathrm{Mod}_R(G,I)$*.*

3) The I*-invariant functor*

$$V \to V^I : \mathrm{Mod}_R(G,I) \to \mathrm{Mod}H_R(G,I)$$

is an equivalence of categories.

4) The I^ℓ*-invariant functor on the unipotent block* $\mathcal{B}_{R,1}(G)$

$$V \to V^{I^\ell} : \mathcal{B}_{R,1}(G) \to \mathrm{Mod}H_R(G,I^\ell)$$

is an equivalence of categories.

5) In the banal case, $\mathrm{Mod}_R(G,I)$ *is the unipotent block.*

6) In the limit case, $\mathrm{Mod}_R(G,I)$ *is not the unipotent block.*

7) The parabolically induced representation $\mathrm{Ind}_B^G 1$ *is semi-simple (hence also* $\mathrm{Ind}_P^G 1$ *for all parabolic subgroups* P *of* G*). In the limit case,* $\mathrm{Ind}_B^G \mathbf{X}$ *is semi-simple for any unramified* R*-character* $\mathbf{X} : T/T_o \to R^*$ *of* T*.*

8) In the limit case, the R*-algebra* $H_R(G,I_\ell)$ *is isomorphic to the natural twisted tensor product of* $H_R(G,I)$ *and* $R[I^\ell]$*.*

The proof of the theorem uses some general results (A), ..., (H), valid in the non quasi-banal case (except (E) and (G)) and for most of them when G is a general reductive connected p-adic group. We recall them first.

(A) The algebra $R[T/T_o]$ is identified to its image in $H_R(G,I)$ by the *Bernstein embedding*

$$t_B : R[T/T_o] \to H_R(G,I) \tag{1}$$

such that the U-coinvariants induces a $R[T/T_o]$-isomorphism

$$V^I \simeq (V_U)^{T_o} \tag{2}$$

for any $V \in \mathrm{Mod}_R G$ [V2, II.10.2].

(B) By [Dat], we have a $(G, R[T/T_o])$-isomorphism

$$R[I\backslash G] \simeq \mathrm{Ind}_B^G R[T/T_o] \tag{3}$$

when $R[T/T_o]$ is embedded in $H_R(G,I)$ by the Bernstein embedding $t_{\overline{B}} : R[T/T_o] \to H_R(G,I)$, defined by the opposite (lower triangular) $\overline{B}$ of B as in (A), where $R[T/T_o]$ is the universal representation of T inflated to B.

Hence for any character $\mathbf{X} : T/T_o \to R^*$ i.e. an algebra homomorphism $R[T/T_o] \to R$

$$R \otimes_{\mathbf{X},R[T/T_o],t_{\overline{B}}} R[I\backslash G] \simeq \mathrm{Ind}\,_B^G\mathbf{X} \tag{4}$$

$$R \otimes_{\mathbf{X},R[T/T_o],t_{\overline{B}}} H_R(G,I) \simeq (\mathrm{Ind}\,_B^G\mathbf{X})^I. \tag{5}$$

(C) The compact induction from an open compact subgroup K of G to G has a *right* adjoint the restriction from G to K [V1, I.5.7]. In particular, a representation generated by its I-invariant vectors is a quotient of a direct sum of $R[I\backslash G]$ (denoted $\oplus R[I\backslash G]$).

(D) The double cosets of G modulo (I_p, I) are in bijection with the double cosets of G modulo (I, I). This is clear by the Bruhat decomposition. In particular, the I_p-invariants of $R[I\backslash G]$ is equal to the I-invariants.

(E) In the quasi-banal case, every cuspidal irreducible representation of every Levi subgroup of G is supercuspidal [V1, III.5.14].

(F) The irreducible unipotent representations are the irreducible subquotients of $R[I\backslash G]$ by [V2, IV.6.2].

(G) When $q = 1$ in R, the Iwahori-Hecke algebra is the group algebra of the affine symmetric group

$$N/T_o \simeq W.(T/T_o) \simeq S_n\mathbf{Z}^n$$

(semi-direct product) where N is the normalizer of T in G and $W := N/T$ with its natural action on T/T_o. Naturally $T/T_o \simeq \mathbf{Z}^n$ by choice of a uniformising parameter p_F of F and $W \simeq S_n$ the symmetric group on n letters with its natural action on $\mathbf{Z}^n$. The natural embedding

$$R[T/T_o] \to H_R(G,I) \simeq R[W.(T/T_o)] \tag{6}$$

is equal to $t_B = t_{\overline{B}}$. These properties are deduced without difficulty from [V1, I.3.14], [V2, II.8].

(H) When $q = 1$ in R, let π_i be an irreducible R-representation of the group $GL(n_i d_i, F)$ with cuspidal support $\otimes^{n_i}\sigma_i$, for an irreducible cuspidal R-representation σ_i of $GL(d_i, F)$ for all $1 \le i \le k$. Suppose that σ_i is not equivalent to σ_j if $i \neq j$. Then the representation of $GL(\sum_i n_i d_i, F)$ parabolically induced from $\pi_1 \otimes \ldots \otimes \pi_k$ is irreducible by [V2, V.3].

Proof of the theorem 1 We suppose that we are in the quasi-banal case.

a) We prove that any irreducible subquotient V of $R[I\backslash G]$ has a non zero I-invariant vector. The U-coinvariants V_U of any irreducible subquotient V of the representation (3) have a non zero vector invariant by T_o, by (E). By (2), V has a non zero I-invariant vector.

b) We prove that if $W \subset V$ are subrepresentations of $\oplus R[I\backslash G]$, then $W^I = V^I$ implies $W = V$, and $V^I = V^{I_p}$. The geometric property (D) implies that the I_p-invariants of any subrepresentation of $\oplus R[I\backslash G]$ is equal to its I-invariants. Hence $W^I = W^{I_p}, V^I = V^{I_p}$. The functor of I_p-invariants is exact and any irreducible subquotient of $R[I\backslash G]$ has a non zero I_p-invariant vector by a). Hence $W^{I_p} = V^{I_p}$ implies $W = V$.

c) We prove the property 1) of the theorem. The property is trivial with quotient instead of subquotient. Let $Y \subset X$ and $p : \oplus R[I\backslash G] \to X$ a surjective G-homomorphism. Let us denote by V the inverse image of Y by p, and by W the subrepresentation of V generated by V^I. We have $W^I = V^I$ by construction, hence $W = V$ by b). Hence V is generated by its I-invariant vectors. The same is true for its quotient Y.

d) We prove the property 2) of the theorem. In c) V is a subrepresentation of $\oplus R[I\backslash G]$ hence we have $V^I = V^{I_p}$ by b). The functor of I_p-invariants is exact hence $p(V^{I_p}) = Y^{I_p}$. As $Y^I \subset Y^{I_p}$ and $p(V^I) \subset Y^I$ we have $Y^I = Y^{I_p} = p(V^I)$. This is valid for any Y hence for any representation of $\mathrm{Mod}_R(G, I)$.

e) We prove the property 3) of the theorem. All the conditions of the theorem of Arabia [A, th.4 2) (b-2)] are satisfied.

f) We prove the property 4) of the theorem. Let V be a unipotent representation. Then V is generated by V^{I^ℓ} by (F). The irreducible subquotients of the action of I on V^{I^ℓ} are trivial, because I/I^ℓ is an ℓ-group. Conversely let V be a representation generated by V^{I^ℓ}. Then the irreducible subquotients of V are unipotent, and a representation such that all its irreducible subquotients are unipotent is unipotent. As the pro-order of I^ℓ is invertible in R, and the unipotent block is generated by $\mathrm{Ind}_{I^\ell}^G 1_R = R[I^\ell\backslash G]$, the I^ℓ-invariant functor is an equivalence of category with the Hecke algebra $H_R(G, I^\ell)$.

g) We prove the property 5) of the theorem. In the banal case $I = I^\ell$ and compare the properties 3) and 4) of the theorem.

h) We prove the property 6) of the theorem. In the limit case, $I \neq I^\ell$. The I-invariants of $\mathrm{Ind}_{I^\ell}^G 1$ can be computed using the decomposition of the parahoric restriction-induction functor [V3, C.1.4] and the simple property

$$\dim(\mathrm{Ind}_{I^\ell}^I 1)^I = 1.$$

One finds that the I-invariants of $\mathrm{Ind}_{I^\ell}^G 1$ are the I-invariants of its proper subrepresentation $\mathrm{Ind}_I^G 1 = R[I\backslash G]$. Hence the unipotent representation $\mathrm{Ind}_{I^\ell}^G 1$ is not generated by its I-invariant vectors.

i) We prove the property 7) of the theorem. In the banal case $\mathrm{Ind}_B^G 1$ is irreducible. We suppose that we are in the limit case. By (4), $\mathrm{Ind}_B^G 1$ is generated by its I-invariant vectors. Hence by the property 3) of the theorem, $\mathrm{Ind}_B^G 1$ is semi-simple if $(\mathrm{Ind}_B^G 1)^I$ is a semi-simple right $H_R(G,I)$-module. By (5) for the trivial character of T, we have

$$(\mathrm{Ind}_B^G 1)^I \simeq R \otimes_{1,R[T/T_o],t_{\overline{B}}} H_R(G,I).$$

By (6), the action of $H_R(G,I) \simeq R[W.(T/T_o)]$ on $(\mathrm{Ind}_B^G 1)^I$ restricted to $R[T/T_o]$ is trivial. As $R[W]$ is semi-simple, $(\mathrm{Ind}_B^G 1)^I$ is a semi-simple right $H_R(G,I)$-module.

Every parabolic subgroup of G is conjugate to a parabolic group P which contains B, and the isomorphism class of $\mathrm{Ind}_P^G 1$ does not change when P is replaced by a conjugate in G. We have an inclusion $\mathrm{Ind}_P^G 1 \subset \mathrm{Ind}_B^G 1$ in $\mathrm{Mod}_R G$. As $\mathrm{Ind}_B^G 1$ is semi-simple, the same is true for $\mathrm{Ind}_P^G 1$.

Let $\mathbf{X}$ be an unramified R-character of T. Modulo conjugaison $\mathbf{X} = \otimes_i \mathbf{X}_i$ is the external product of characters $\mathbf{X}_i := x_i 1$ of the diagonal subgroups T_i of $G_i := GL(n_i, F)$, which are different multiples of the identity character, $x_i \neq x_j \in R^*$ if $i \neq j$ and $\sum_i n_i = n$. The parabolic induction $\mathrm{Mod}_R \prod_i G_i \to \mathrm{Mod}_R G$ sends any irreducible subquotient of $\otimes_i \mathrm{Ind}_{B_i}^G x_i 1$ to an irreducible representation of G by (H). This implies the semi-simplicity of $\mathrm{Ind}_B^G \mathbf{X}$.

j) We prove the property 8). Let V be an R-vector space with an action $\sigma : I \to GL_R(V)$ of I trivial on I_p. We have $I = T_o I_p$. The Weyl group $W \simeq S_n$ embedded in G as usual, acts on T_o by conjugation. By inflation, the affine Weyl group $W.(T/T_o)$ acts on T_o. For $w \in w_o.(T/T_o)$ with $w_o \in W$, one denotes by $\mathrm{Int}w.V$ the space V with the action of I such that $k \in t_o I_p$ acts by $\sigma(w_o t_o w_o^{-1})$ for $t_o \in T_o$. The endomorphism algebra $\mathrm{End}_{RG}\mathrm{Ind}_I^G V$ is isomorphic as an R-module to ([V2, II.2 page 562] and [V3, C.1.5]):

$$\mathrm{End}_{RG}\mathrm{Ind}_I^G V \simeq \oplus_{w \in W.(T/T_o)} \mathrm{Hom}_{RI}(V, \mathrm{Int}w.V). \tag{8}$$

A function in $\mathrm{Ind}_I^G V$ with support Ig and value $v \in V$ at $g \in G$ is denoted by $[Ig, v]$. We have $g^{-1}[I, v] = [Ig, v]$. The endomorphism $T_{w,A}$ corresponding to $w \in W.(T/T_o), A \in \mathrm{Hom}_{RI}(V, \mathrm{Int}w.V)$ in (8) is defined by [V2, II.2, page 562]:

$$T_{w,A}[I, v] = \sum_{x \in (I_p \cap w^{-1} I_p w)\backslash I_p} [Iwx, A(v)] = \sum_{x \in (I_p \cap w^{-1} I_p w)\backslash I_p} (wx)^{-1}[I, A(v)] \tag{9}$$

because $IwI = \cup_{x\in(I_p\cap w^{-1}I_p w)\backslash I_p} Ix$ is a disjoint decomposition and I_p acts trivially on V. The product in $\mathrm{End}_{RG}\mathrm{Ind}_I^G V$ is given by

$$T_{w',A'}T_{w,A}[I,v] = \sum_{x\in(I_p\cap w^{-1}I_p w)\backslash I_p, y\in(I_p\cap(w')^{-1}I_p w')\backslash I_p} (wx)^{-1}(w'y)^{-1}[I,(A'\circ A)(v)],$$

or equivalently,

$$T_{w',A'}T_{w,A}[I,v] = \sum_{x\in(I_p\cap w^{-1}I_p w)\backslash I_p, y\in(I_p\cap(w')^{-1}I_p w')\backslash I_p} [Iw'ywx,(A'\circ A)(v)]. \tag{10}$$

The Iwahori-Hecke algebra $H_R(G,I)$ is the R-algebra of RG-endomorphisms of $\mathrm{Ind}_I^G 1_R$. We denote T_w for $T_{w,\mathrm{Id}}$ in $H_R(G,I)$. The Hecke algebra $H_R(G,I^\ell)$ is the R-algebra of RG-endomorphisms of $\mathrm{Ind}_I^G V$ where $V = R[I/I^\ell]$ with the regular action σ of I. Let i_w be the R-linear automorphism of $V \simeq k[I_\ell]$ given by conjugation by $w\in W.(T/T_o)$. The R-linear map $A\mapsto i_w\circ A$ from $\mathrm{End}_{RI}(V)$ to $\mathrm{Hom}_{RI}(V,\mathrm{Int}w.V)$ is an isomorphism. We have $T_{w,i_w\circ A} = T_{w,i_w}T_{1,A}$ in $H_R(G,I^\ell)$ and the R-linear map defined by

$$T_w\otimes A\mapsto T_{w,i_w}T_{1,A} : H_R(G,I)\otimes_R \mathrm{End}_{RI}(V)\mapsto H_R(G,I^\ell)$$

is an isomorphism. The injective R-linear map $A\mapsto T_{1,A} : \mathrm{End}_{RI}(V)\to H_R(G,I^\ell)$ respects the product. In the limit case, the injective R-linear map such that $T_w\mapsto T_{w,i_w} : H_R(G,I)\to H_R(G,I^\ell)$ respects also the product because $T_{w'}T_w = T_{w'w}$ in $H_R(G,I)$ and $T_{w',A'}T_{w,A} = T_{w'w,i_w^{-1}\circ A'\circ i_w\circ A}$ in $H_R(G,I^\ell)$. We have $\mathrm{End}_{RI}V = \mathrm{End}_{RI_\ell}V = R[I_\ell]$. $\diamond$

Let $\mathcal{J}_R$ *be the annihilator of* $R[G/I]$. The Schur R-algebra of G is Morita equivalent to $\mathcal{H}_R(G)/\mathcal{J}_R$ [V3, 2]. It is clear that $\mathcal{J}_R$ annihilates the abelian category $\mathrm{Mod}_R(G,I)$.

2 Theorem *In the quasi-banal case, the category* $\mathrm{Mod}_R(G,I)$ *is the category of representations of G which are annihilated by $\mathcal{J}_R$. In other terms, the Schur R-algebra of G is Morita equivalent to the Iwahori-Hecke R-algebra of G.*

This is already known in the banal case. The proof of the theorem results from properties of the Gelfand-Graev representation Γ_R and of the Steinberg representation St_R of $GL(n,\mathbf{F}_q)$.

We need more notation.

a) The subcategory $\mathrm{Mod}_{R,1}GL(n,\mathbf{F}_q)$ of $\mathrm{Mod}_R GL(n,\mathbf{F}_q)$ generated by (the irreducible subquotients of) $R[GL(n,\mathbf{F}_q)/B(\mathbf{F}_q)]$ is a sum of blocks by a theorem of Broué-Malle. Representations in $\mathrm{Mod}_{R,1}GL(n,\mathbf{F}_q)$ are called *unipotent*. The annihilator $\mathcal{J}_R(q)$ of $R[GL(n,\mathbf{F}_q)/B(\mathbf{F}_q)]$ in $R[GL(n,\mathbf{F}_q)]$ is the Jacobson radical of the unipotent part of the group algebra $R[GL(n,\mathbf{F}_q)]$, because the representation $R[GL(n,\mathbf{F}_q)/B(\mathbf{F}_q)]$ is semi-simple.

b) Let $\psi : \mathbf{F}_q \to R^*$ be a non trivial character. We extend ψ to a character $(u_{i,j}) \to \psi(\sum u_{i,i+1})$ of the strictly upper triangular subgroup $U(\mathbf{F}_q)$ of $GL(n,\mathbf{F}_q)$, still denoted by ψ. The representation of $GL(n,\mathbf{F}_q)$ induced by the character ψ of $U(\mathbf{F}_q)$ is the *Gelfand-Graev representation* $\mathit{\Gamma}_R$. Its isomorphism class does not depend on ψ. We denote by $\mathit{\Gamma}_{R,1}$ the unipotent part of $\mathit{\Gamma}_R$.

c) The Steinberg representation St_R of $GL(n,\mathbf{F}_q)$ is the unique irreducible R-representation such that, as a right module for the Hecke algebra $H_R(GL(n,\mathbf{F}_q),B(\mathbf{F}_q))$, its module of $B(\mathbf{F}_q)$-invariants is isomorphic to the sign representation.

d) The inflation followed by the compact induction is an exact functor

$$i^G : \mathrm{Mod}_R GL(n,\mathbf{F}_q) \to \mathrm{Mod}_R GL(n,O_F) \to \mathrm{Mod}_R G$$

e) The global Hecke algebra $\mathcal{H}_R(G)$ contains the Hecke algebra

$$\mathcal{H}_R^o := H_R(GL(n,O_F), 1 + p_F M(n,O_F))$$

isomorphic via inflation to the group algebra $R[GL(n,\mathbf{F}_q)]$. The Jacobson radical $\mathcal{J}_R(q)$ of the unipotent part of the group algebra $R[GL(n,\mathbf{F}_q)]$ identifies with a two-sided ideal of $\mathcal{H}_R^o$.

We recall [V3, theorem 4.1.4]:

(I) The representation of $GL(n,\mathbf{F}_q)$ on the $1+p_F M(n,O_F)$-invariants of $R[G/I]$ is isomorphic to a direct sum $\oplus R[GL(n,\mathbf{F}_q)/B(\mathbf{F}_q)]$.

(J) $i^G V$ is generated by its I-invariant vectors if $V \in \mathrm{Mod}_R GL(n,\mathbf{F}_q)$ is generated by its $B(\mathbf{F}_q)$-invariant vectors.

4 Lemma *Suppose that we are in the quasi-banal case. Then*

1) $\mathcal{J}_R$ is the Jacobson radical of the unipotent bloc of $\mathrm{Mod}_R G$ (same for $\mathcal{J}_R(q)$ and $GL(n,\mathbf{F}_q)$).

2) The unipotent part $\mathit{\Gamma}_{R,1}$ of the Gelfand-Graev R-representation of the group $GL(n,\mathbf{F}_q)$ is the projective cover of the Steinberg R-representation St_R of $GL(n,\mathbf{F}_q)$.

3) $\mathit{\Gamma}_{R,1}\mathcal{J}_R(q)$ is the kernel of the map $\mathit{\Gamma}_{R,1} \to \mathrm{St}_R$.

4) $\mathcal{J}_R(q) \subset \mathcal{J}_R$.

5) $i^G\mathit{\Gamma}_{R,1}/(i^G\mathit{\Gamma}_{R,1})\mathcal{J}_R$ *is a quotient of* $i^G\mathrm{St}_R$ *and is generated by its* I*-invariant vectors.*

Proof of the lemma This is known in the banal case, hence we suppose that we are in the limit case.

We prove the property 1). The semi-simplicity of $\mathrm{Ind}_B^G\mathbf{X}$ for all unramified characters (theorem 1 7)) implies with (3) that $\mathcal{J}_R$ *is the Jacobson radical of the unipotent bloc*. This means that $\mathcal{J}_R$ is the intersection of the annihilators in the global Hecke algebra $\mathcal{H}_R(G)$ of the irreducible unipotent R-representations of G.

We prove the property 2). The induced representation $\mathrm{Ind}_{B(\mathbf{F}_q)}^{GL(n,\mathbf{F}_q)}1_R$ is semi-simple, and St_R is the *unique* subquotient which is isomorphic to a quotient of the Gelfand-Graev representation $\mathit{\Gamma}_R$. By the *uniqueness* theorem,

$$\dim_R \mathrm{Hom}_{RG}(\mathit{\Gamma}_R, \mathrm{St}_R) = 1.$$

The unipotent part $\mathit{\Gamma}_{R,1}$ of the Gelfand-Graev representation $\mathit{\Gamma}_R$ is projective (because the characteristic of R is different from p) and is a direct sum of indecomposable projective representations of $GL(n,\mathbf{F}_q)$. In the quasi-banal case, the two properties of uniqueness imply that $\mathit{\Gamma}_{R,1}$ is projective cover of St_R.

The property 3) results from 1) and 2) by general results [CRI 18.1].

The property 4) results from e) and (I).

We prove the property 5). By definition $(i^G\mathit{\Gamma}_R)\mathcal{J}_R = \mathit{\Gamma}_R \otimes_{\mathcal{H}_R^o} \mathcal{J}_R$.

By 4) $\mathit{\Gamma}_R \otimes_{\mathcal{H}_R^o} \mathcal{J}_R(q)\mathcal{H}_R(G) \subset \mathit{\Gamma}_R \otimes_{\mathcal{H}_R^o} \mathcal{J}_R$.

We have [V1 I.5.2.c)] $\mathit{\Gamma}_R \otimes_{\mathcal{H}_R^o} \mathcal{J}_R(q)\mathcal{H}_R(G) = \mathit{\Gamma}_R\mathcal{J}_R(q) \otimes_{\mathcal{H}_R^o} \mathcal{H}_R(G) = i^G W$ where $W = \mathit{\Gamma}_R\mathcal{J}_R(q)$. Clearly $i^G\mathit{\Gamma}_R/(i^G\mathit{\Gamma}_R)\mathcal{J}_R$ is a quotient of $i^G\mathit{\Gamma}_R/i^G W$.

The functor i^G is exact hence $i^G\mathit{\Gamma}_R/i^G W \simeq i^G(\mathit{\Gamma}_R/W)$. By 3) $\mathit{\Gamma}_R/W \simeq \mathrm{St}_R$. Hence $i^G\mathit{\Gamma}_R/(i^G\mathit{\Gamma}_R)\mathcal{J}_R$ is a quotient of $i^G\mathrm{St}_R$. By c), St_R is irreducible and has a non zero vector invariant by $B(\mathbf{F}_q)$. By (J), $i^G\mathrm{St}_R$ is generated by its I-invariant vectors. $\diamond$

Lemma 4 extends to the standard Levi subgroups $M_\lambda(\mathbf{F}_q)$ of $GL(n,\mathbf{F}_q)$, quotients of the parahoric subgroup $P_\lambda(O_F)$. These groups are parametrised by the partitions λ of n. The group $GL(n,\mathbf{F}_q)$ corresponds to the partition (n). One denotes by an index λ the objects relative to λ.

We recall:

(K) $\mathbf{Q_R} := \mathbf{\Gamma_R}/\mathbf{\Gamma_R}\mathcal{J}_R$ is a projective generator of $\mathrm{Mod}\mathcal{H}_R(G)/\mathcal{J}_R$ where $\mathbf{\Gamma_R} := \oplus_\lambda i_\lambda^G \mathit{\Gamma}_{R,\lambda}$ [V3, theorem 5.13].

Proof of the theorem 3 By lemma 4 for the group $M_\lambda(\mathbf{F}_q)$, the quotient $i_\lambda^G\mathit{\Gamma}_{R,\lambda}/i_\lambda^G\mathit{\Gamma}_{R,\lambda}\mathcal{J}_R$ of $i_\lambda^G\mathrm{St}_{R,\lambda}$ is generated by its I-invariant vectors. Hence the progenerator $\mathbf{Q_R}$ of Mod$\mathcal{H}_R(G)/\mathcal{J}_R$ is generated by its I-invariant vectors. $\diamond$

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